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The homology groups of small cover on a triangular prism and its number of characteristic functions

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Abstract: Triangular prism is a common geometric shape. From the perspective of algebraic topology, it is a familiar simple convex polyhedron in algebraic topology. In this paper, we mainly calculate that there are only two kinds of characteristic functions on a triangular prism, and the homology groups of triangular prism is obtained by different characteristic functions are different. Firstly, according to the Morse function on the convex polytope P^n, We can give the cell decoposition of the corresponding small cover M^n over P^n, and the cellular chain complex $\{D_i(M^n(\lambda)),\partial_i\}$ of M^n. Secondly, considering the relationship between the boundary homomorphism $\{\partial_i\}$ and the characteristic function λ, we can give the principle of how to determine the boundary homomorphism is given. Finally, the homology groups are computed by defination $\{H_i = \ker\partial_i / \mathrm{Im}\partial_{i+1}\}$, we can give the corresponding results.

1. Foreword

The notion of small covers was firstly introduced by M. Davis and T. Jnuszkiewicz in [1]. An n-dimensional small cover is n-dimensional smooth manifold M^n adimtting a Z_2^n-action and its orbit space is an n-dimensional simple convex polytope. We can construct a small cover $M^n(P^n,\lambda)$ over P^n, and λ is the characteristic function. Regarding the calculation of the homology of small cover, literature [2] mainly discusses the flexibility coefficient. Literature [3] was calculated using the cellular chain complex. Many further works has been carried out (see in [4-10]). In this paper, the reference [3] cellular chain complex is used to calculate the homology group of small cover on the Triangular prism, and the corresponding results are obtained.

2. Prepare knowledge

Definition 2.1 [1] Setting P^n is an n-dimensional convex polyhedron, and if each vertex of P^n has n codimensions of 1, we call this convex polyhedron is single convex polyhedron.

Definition 2.2 [1] Function $\lambda : F \to Z_2^n$ is called characteristic function, if you meet the following conditions: any the vertices, have $|\lambda(F_1),\lambda(F_2),.....,\lambda(F_n)| = \pm 1$.

Definition 2.3 [1] (Morse function) The Morse function on a single convex polyhedron P^n refers to a high flow function on a certain dimensional skeleton. Denoted $ind(v)$ as the number of flow directions at the vertex v on a single convex polyhedron P^n .Setting $h_i = |\{v|ind(v)=i\}|$,

$$h(P^n)=(b_1,b_2,...,b_n).$$

Lemma 2.1 [1] $\pi : M^n \to P^n$ is a small cover on single convex polyhedra P^n, setting $b_i = \dim H_i(M^n, Z_2^n)$,then $h(P^n)=(h_0,h_1,...h_n)=(b_1,b_2,...,b_n)$. Let $M^n(\lambda)$ be the single convex polyhedron P^n on the small cover of the characteristic function λ, we can write out on the cell chamber chain complex as:

$$0 \to D_n(M^n(\lambda)) = Z(v) \xrightarrow{\partial_n} D_{n-1}(M^n(\lambda)) = \bigoplus_{ind(v')=h_{n-1}} Z(v) \xrightarrow{\partial_{n-1}} .... \xrightarrow{\partial_2} D_1(M^n(\lambda)) = \bigoplus_{h_1} Z \xrightarrow{\partial_1 = 0} D_0(M^n(\lambda)) = Z \to 0$$

Lemma 2.2 [3] set $M^n(\lambda)$is the small cover on a monoconvex polyhedron P^n with characteristic function λ, and the complex of the cellular chain is $\{D_i(M^n(\lambda)), \partial_i\}$. We have the following conclusions:

(1) $\partial_n : Z(v) \to \bigoplus_{v' \in I(n-1)} Z(v')$ In the following forms: $\partial_n : x \to (..., a_{v'}x, ...)$ $a_{v'} \in \{0, 2\}$.

Let $\lambda(F_{v'}) = (a_1, ...a_n)$, if $a_1 + ... + a_n$ the sum is odd, then $a_{v'} = 0$; if $a_1 + ... + a_n$ the sum is even $a_{v'} = 2$.

(2) $\{\partial_q\}_{1 \le q \le n} : \bigoplus_{v \in I(q)} Z(v) \to \bigoplus_{v' \in I(q-1)} Z(v')$ An element above has the following form:

$\partial_q : x \to (..., a_{v'}x, ...)$ $a_{v'} \in \{0, \pm 2\}$. If $F_{v'} \not\subset F_v$, then there is $a_{v'} = 0$.

3. Homology group of small cover on the triangular prism

3.1 Combined structure of triangular prism and the cellular chain complex

The triangular prism is composed of five faces, and the five faces are e_1, e_2, e_3, e_4, e_5.Figure 1 below is the height flow function (Morse function) on the triangular prism, calculating the index at each vertex:

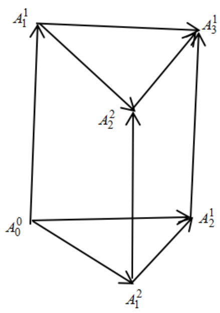

Figure 1: Height flow function on three prism (Morse function)

$$ind(A_0^0) = 0, ind(A_1^1) = ind(A_1^2) = 1,$$

$$ind(A_2^1) = ind(A_2^2) = 2, ind(A_3^1) = 3.$$

On the prism satisfies $v = F_1 \cap F_2 \cap ... \cap F_5$ any of the vertices of the triangular prism

$\left|\lambda(F_1),\lambda(F_2),\ldots\ldots,\lambda(F_n)\right|=\pm 1$, And the determinant value of the characteristic function of any two edges at any vertex is ± 1. There are only two characteristic functions of the five faces on the three prism:

The first: $\lambda(e_1)=(1,0,0)$, $\lambda(e_2)=(0,1,0)$, $\lambda(e_3)=(0,0,1)$, $\lambda(e_4)=(1,1,1)$, $\lambda(e_5)=(0,1,1)$

The second: $\lambda(e_1)=(1,0,0)$, $\lambda(e_2)=(0,1,0)$, $\lambda(e_3)=(0,0,1)$, $\lambda(e_4)=(1,0,1)$, $\lambda(e_5)=(0,1,1)$.

Note: These two characteristic function are calculated in this article.

Cellular chain complex on the triangular prism:

Let be small cover of the characteristic function of the triangular prism, then the complex of the cellular chain complex is:

$$0\to C_3(M^3(\lambda))\xrightarrow{\partial_3}C_2(M^3(\lambda))\xrightarrow{\partial_2}C_1(M^3(\lambda))\xrightarrow{\partial_1=0}C_0(M^3(\lambda))\to 0$$

among $C_3(M^3(\lambda))=Z(A_3^1)$, $C_2(M^3(\lambda))=\oplus_{i=1}^2 Z(A_2^i)$, $C_1(M^3(\lambda))=\oplus_{i=1}^2 Z(A_1^i)$, $C_0(M^3(\lambda))=Z(A_0^0)$

The expression of the generating element is given as follows:

$Z(A_2^1)$represent $\pi^{-1}(e_4)=\pi^{-1}(A_0^0A_1^2A_2^1)$, $Z(A_2^2)$represent $\pi^{-1}(e_5)=\pi^{-1}(A_1^1A_0^0A_1^2A_2^2)$

$$Z(A_1^1)=\pi^{-1}(A_0^0A_1^1), Z(A_1^2)=\pi^{-1}(A_0^0A_1^2).$$

3.2 Proof of the main conclusions

The first characteristic function of the five faces of the triangular prism:

$$\lambda(e_1)=(1,0,0),\ \lambda(e_2)=(0,1,0),\ \lambda(e_3)=(0,0,1),\ \lambda(e_4)=(1,1,1),\ \lambda(e_5)=(0,1,1).$$

The homology group of small cover corresponding to the first characteristic function on the triangular prism is:

$$H_0\cong\frac{Ker\partial_0}{\operatorname{Im}\partial_1}\cong Z(A_0^0)=Z,$$

$$H_1\cong\frac{Ker\partial_1}{\operatorname{Im}\partial_2}\cong Z_2(A_1^1)\oplus Z(A_1^2),$$

$$H_2\cong\frac{Ker\partial_2}{\operatorname{Im}\partial_3}\cong Z(A_2^1)\oplus Z_2(A_2^2),$$

$$H_3\cong\frac{Ker\partial_3}{\operatorname{Im}\partial_4}\cong Z(A_3^1)=Z$$

Proof: Homology group under type I characteristic function:

Step 1: calculate the characteristic function of the edges, for example e_4, the specific calculation process is as follows:

Given the Morse flow of e_4 and surfaces that intersect with e_4, as shown in Figure 2 below:

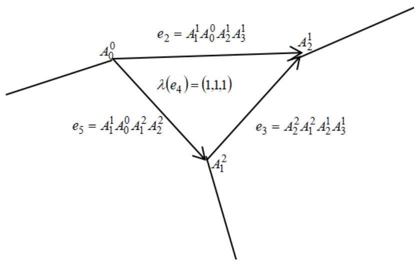

Figure 2: $e_4 = A_0^0 A_1^2 A_2^1$

Then the three edges on e_4 are: $A_0^0A_1^2 = e_5 \cap e_4, A_0^0A_2^1 = e_2 \cap e_4, A_1^2A_2^1 = e_3 \cap e_4$.

Second, there are unique vertices A_2^1 on e_4 which make $ind(A_2^1) = 2$ and $A_2^1 = A_0^0A_2^1 \cap A_1^2A_2^1$, without loss of generality, Order $\lambda(A_0^0A_2^1) = (0,1), \lambda(A_1^2A_2^1) = (1,0)$ is the standard base Z_2^2 , marking the characteristic function of e_4, the three surfaces $e_{2,} e_{3,} e_5$ and the standard base of the edges, as shown in Figure 3 below:

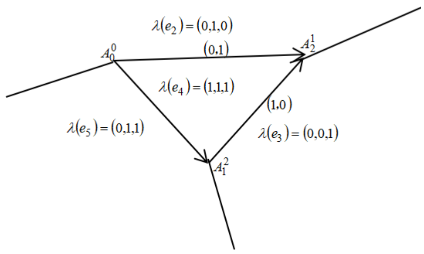

Figure 3: Characteristic function of the surfaces $e_{2,} e_{3,} e_5$

Finally, the characteristic function of each upper edge on e_4 is calculated as follows:

Because $ind(A_2^1) = 2$ is the highest point of the flow direction of the vertex e_4 on the surface, it is taken A_2^1 as the base point, at this point we have: $A_0^0A_2^1 = e_2 \cap e_4, A_1^2A_2^1 = e_3 \cap e_4$.

Make a homomorphism as follows: $\lambda(e_4) = (1,1,1) \to (0,0), \lambda(e_2) = (0,1,0) \to (0,1), \lambda(e_3) = (0,0,1) \to (1,0)$.

Owing to $A_0^0A_2^1 = e_2 \cap e_4$, have $\lambda(e_2) = (0,1,0) \to (0,1) = \lambda(A_0^0A_2^1)$

Owing to $A_1^2A_1^2 = e_3 \cap e_4$, have $\lambda(e_3) = (0,0,1) \to (1,0) = \lambda(A_1^2A_2^1)$

Owing to $A_0^0A_1^2 = e_5 \cap e_4$, have $\lambda(e_5) = (0,1,1) \to (0,1,0) + (0,0,1) \to (0,1) + (1,0) \to (1,1) = \lambda(A_0^0A_1^2)$

From the above calculation results know the characteristic function of each edge on e_4 , as shown in Figure 4 below:

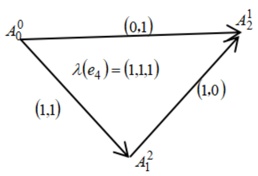

Figure 4: Characteristic function of each edge on the surface e_4

According to the previous calculation method, each side on the surface e_5 is marked on the graph, as shown in Figure 5 below:

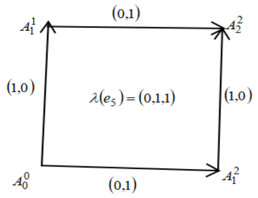

Figure 5: Characteristic function of each edge on the surface e_5

Step 2: Calculate the edge homomorphism $\partial_q(q=2,3)$

(1) Calculation of ∂_3:

So $x \in \lambda(Z(A_3^1))$, because $\lambda(Z(A_2^1)) = \lambda(e_4) = (1,1,1)$, and $1+1+1=3$ For odd numbers, $\lambda(Z(A_2^2)) = \lambda(e_5) = (0,1,1)$, and $0+1+1=2$ For even numbers.

(2) the calculation of the ∂_2

Order $x_1 \in \lambda(Z(A_2^1))$, because $x_1 \in \lambda(Z(A_2^1))$ and $1+1=2$ is an even number.

From the conclusions of Lemma 2.2: $\partial_2 : x_1 \to (0x_1, \pm 2x_1)$

in like manner $\partial_2 : x_2 \to (0x_2, 0x_2)$

In summary, collated into the form of a matrix.

$$\partial_3 : x \to (0x, \pm 2x)$$

$$\partial_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0x_1 + 0x_2 \\ \pm 2x_1 + 0x_2 \end{pmatrix}.$$

Step 3: Calculation of the homology group:

Cellular chain complex on a triangular prism:

Let $M^n(\lambda)$ be small cover of the characteristic function λ of the triangular prism, then the complex of the cellular chain complex is: $0 \to C_3(M^3(\lambda)) \xrightarrow{\partial_3} C_2(M^3(\lambda)) \xrightarrow{\partial_2} C_1(M^3(\lambda)) \xrightarrow{\partial_1=0} C_0(M^3(\lambda)) \to 0$

The homology group of the corresponding stained small cover $M(\lambda)^3$ on the triangular prism are:

$$H_0 \cong \frac{Ker\partial_0}{\operatorname{Im}\partial_1} \cong Z(A_0^0) = Z\ ,$$

$$H_1 \cong \frac{Ker\partial_1}{\operatorname{Im}\partial_2} \cong Z_2(A_1^1) \oplus Z(A_1^2),$$

$$H_2 \cong \frac{Ker\partial_2}{\operatorname{Im}\partial_3} \cong Z(A_2^1) \oplus Z_2(A_2^2),$$

$$H_3 \cong \frac{Ker\partial_3}{\operatorname{Im}\partial_4} \cong Z(A_3^1) = Z\ \cdot$$

among,

$$\ker\partial_1 = Z(A_1^1) \oplus Z(A_1^2),$$

$$\ker\partial_2 = Z(A_2^1) \oplus Z(A_2^2),$$

$$\ker\partial_3 = Z\left(A_3^1\right),$$

$$\operatorname{Im}\partial_2 = 2Z\left(A_1^1\right),$$

$$\operatorname{Im}\partial_3 = 2Z\left(A_2^2\right)\ .$$

Following the above calculation method, the second characteristic function is used to calculate the homology group of small cover on the triangular prism:

The second type of staining:

$$\lambda(e_1)=(1,0,0),\lambda(e_2)=(0,1,0),\lambda(e_3)=(0,0,1),\lambda(e_4)=(1,0,1),\lambda(e_5)=(1,1,1)$$

According to the previous calculation method, the characteristic function of each side on the graph as shown in Figure 6:

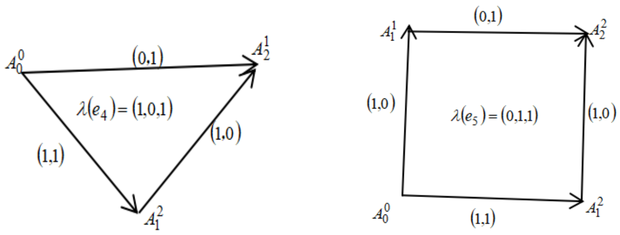

Figure 6: Characteristic function of each edge on the surface e_4, e_5

Edge homomorphism was calculated according to the previous calculation method $\partial_q(q=2,3)$:

$$\partial_3 : x \to (\pm 2x, \mp 2x)$$

$$\partial_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0x_1 + 0x_2 \\ \pm 2x_1 \mp 2x_2 \end{pmatrix}$$

Corresponding corresponding stained λ small cover on the triangular prism, the homology group is,

$$H_0 \cong \frac{Ker\partial_0}{\operatorname{Im}\partial_1} \cong Z\left(A_0^0\right) = Z,$$

$$H_1 \cong \frac{Ker\partial_1}{\operatorname{Im}\partial_2} \cong Z_2\left(A_1^1\right) \oplus Z_2\left(A_1^2\right),$$

$$H_2 \cong \frac{Ker\partial_2}{\operatorname{Im}\partial_3} \cong Z_2\left(A_2^1\right) \oplus Z_2\left(A_2^2\right),$$

$$H_3 \cong \frac{Ker\partial_3}{\operatorname{Im}\partial_4} \cong Z\left(A_3^1\right) = Z\quad .$$

among,

$$\ker\partial_1 = Z\left(A_1^1\right) \oplus Z\left(A_1^2\right),$$

$$\ker\partial_2 = Z\left(A_2^1\right) \oplus Z\left(A_2^2\right),$$

$$\ker\partial_3 = Z\left(A_3^1\right),$$

$$\operatorname{Im}\partial_2 = 2Z\left(A_1^1\right)\oplus 2Z\left(A_1^2\right),$$
$$\operatorname{Im}\partial_3 = 2Z\left(A_2^1\right)\oplus 2Z\left(A_2^2\right)$$

conclusion:

Meet any vertex $v = F_1 \cap F_2 \cap F_3 \cap F_4 \cap F_5$ on the triangular prism, $\left|\lambda(F_1), \lambda(F_2), \ldots\ldots, \lambda(F_n)\right| = \pm 1$, after calculating the determinant value of the characteristic function of any two edges that intersect at any vertex is positive 1 or negative 1. There are only two characteristic function of the five faces on the triangular prism, and the results of the homology group of small cover on the triangular prism are not consistent.

The first: $\lambda(e_1)=(1,0,0)$, $\lambda(e_2)=(0,1,0)$, $\lambda(e_3)=(0,0,1)$, $\lambda(e_4)=(1,1,1)$, $\lambda(e_5)=(0,1,1)$.

The second: $\lambda(e_1)=(1,0,0)$, $\lambda(e_2)=(0,1,0)$, $\lambda(e_3)=(0,0,1)$, $\lambda(e_4)=(1,0,1)$, $\lambda(e_5)=(0,1,1)$

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