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An Optimized Solution of the Basic Solution System of Linear Equations

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Abstract: This article presents an method of making out the optimized solution to the basic system of linear equations.In the process, it only needs to use the properties of simple elementary row transformation and number multiplication vector to directly obtain the basic solution system of the homogeneous linear equation system, and also directly obtain the special solution of the non-homogeneous linear equation system and the corresponding basic solution system of the derived system.

1. Introduction

For the basic solution system of a homogeneous linear equation system with infinitely many solutions, the textbooks [1-4] all use $n-r(A)$ independent vector to solve. Li Yun [5] directly constructed the special solution of the non-homogeneous linear equations and the basic solution system of the derived system. Zhao Yanhui [6] transforms A into the row minimalist form $\begin{bmatrix} I_r & A_{12}^* \\ o & o \end{bmatrix}$ through elementary line transformation, but still need to construct the matrix $\begin{bmatrix} A_{12}^* \\ -I_{n-r} \end{bmatrix}$, the column vector is a basic solution system of the equations. Li Guozhong [7], Han Xinshe [8], etc. Is to use elementary transformation.First reduce $\begin{bmatrix} A^T & I_n \\ -b^T & 0 \end{bmatrix}$ to $\begin{bmatrix} D & P_1^T \\ 0 & P_2^T \\ -b^T & 0 \end{bmatrix}$ by elementary row transformation corresponding to $\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$, proving that the column vector of P_2 is a basic solution system of homogeneous linear equations. Then, through the corresponding to the primary transformation $\begin{bmatrix} D & P_1^T \\ 0 & P_2^T \\ -b^T & 0 \end{bmatrix}$ is reduced to $\begin{bmatrix} I_n & 0 \\ C & 1 \end{bmatrix}$, gaining a particular solution of linear equations x_0. For non-homogeneous, this method requires two specific elementary transformations, which is tedious and difficult to understand and remember. He Fangli [9] used the elementary row transformation to transform the coefficient matrix into a row step matrix, but it cannot directly find the basic solution system. It is still necessary to fill in the blanks to construct the vector in the basic solution system. And the process needs the solver to remember the formula.

2. Optimal solution of basic solution system

Consider the solution of general linear equations (1)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \qquad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{cases} \tag{1}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then the matrix form of the system of equations (1) is $Ax = b$, and the matrix form of the derived system of the system of equations (1) is

$$Ax = 0 \tag{2}$$

When $r(A) = r(A,b) = r < n$, there are infinitely many solutions to equations (1) and (2). Applying elementary row transformation to the augmented matrix $(A \vdots b)$ of equations (1), it can be transformed into the following form:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & k_{1\,r+1} & k_{1\,r+2} & \cdots & k_{1\,n} & d_1 \\ 0 & 1 & \cdots & 0 & k_{2\,r+1} & k_{2\,r+2} & \cdots & k_{2\,n} & d_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & k_{r\,r+1} & k_{r\,r+1} & \cdots & k_{r\,r+1} & d_r \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \tag{3}$$

That is, equations (1) and (4) have the same solution

$$\begin{cases} x_1 = d_1 - k_{1\,r+1}x_{r+1} - k_{1\,r+2}x_{r+2} - \cdots - k_{1\,n}x_n \\ x_2 = d_2 - k_{2\,r+1}x_{r+1} - k_{2\,r+2}x_{r+2} - \cdots - k_{2\,n}x_n \\ \qquad \cdots \\ x_r = d_r - k_{r\,r+1}x_{r+1} - k_{r\,r+2}x_{r+2} - \cdots - k_{r\,n}x_n \end{cases} \tag{4}$$

Where, x_{r+1}, x_{r+2}, $\cdots$, x_n are free variables.Let free variables be arbitrary constants $c_i\left(i = 1, 2 \cdots n - r\right)$, then equation(4)is equivalent to $\begin{cases} x_1 = d_1 - k_{1\,r+1}c_1 - k_{1\,r+2}c_2 - \cdots - k_{1\,n}c_{n-r} \\ x_2 = d_2 - k_{2\,r+1}c_1 - k_{2\,r+2}c_2 - \cdots - k_{2\,n}c_{n-r} \\ \qquad \cdots \\ x_r = d_r - k_{r\,r+1}c_1 - k_{r\,r+2}c_2 - \cdots - k_{r\,n}c_{n-r} \\ x_{r+1} = c_1 \\ x_{r+2} = c_2 \\ \qquad \cdots \\ x_n = c_{n-r} \end{cases}$

or

$$\begin{cases} x_1 = d_1 - k_{1\,r+1}c_1 - k_{1\,r+2}c_2 - \cdots - k_{1\,n}c_{n-r} \\ x_2 = d_2 - k_{2\,r+1}c_1 - k_{2\,r+2}c_2 - \cdots - k_{2\,n}c_{n-r} \\ \qquad \cdots \\ x_r = d_r - k_{r\,r+1}c_1 - k_{r\,r+2}c_2 - \cdots - k_{r\,n}c_{n-r} \\ x_{r+1} = 1c_1 + 0c_2 + \cdots + 0c_{n-r} \\ x_{r+2} = 0c_1 + 1c_2 + \cdots + 0c_{n-r} \\ \qquad \cdots \\ x_n = 0c_1 + 0c_2 + \cdots + 1c_{n-r} \end{cases} \tag{5}$$

the matrix form of equation(5)is equivalent to:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -k_{1\,r+1} \\ -k_{2\,r+1} \\ \vdots \\ -k_{r\,r+1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -k_{1\,r+2} \\ -k_{2\,r+2} \\ \vdots \\ -k_{r\,r+2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_{n-r} \begin{bmatrix} -k_{1\,n} \\ -k_{2\,n} \\ \vdots \\ -k_{r\,n} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{6}$$

or

$$\eta = \begin{bmatrix} d_1 & d_2 & \cdots & d_r & 0 & 0 & \cdots & 0 \end{bmatrix}^T$$

$$\xi_1 = \begin{bmatrix} -k_{1\,r+1} & -k_{2\,r+1} & \cdots & -k_{r\,r+1} & 1 & 0 & \cdots & 0 \end{bmatrix}^T$$

$$\xi_2 = \begin{bmatrix} -k_{1\,r+2} & -k_{2\,r+2} & \cdots & -k_{r\,r+2} & 0 & 1 & \cdots & 0 \end{bmatrix}^T$$

$$\cdots$$

$$\xi_{n-r} = \begin{bmatrix} -k_{1\,n} & -k_{2\,n} & \cdots & -k_{r\,n} & 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

then equation (6) is transformed into

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = \eta + c_1\xi_1 + c_2\xi_2 + \cdots + c_{n-r}\xi_{n-r} \tag{7}$$

Theorem 1: $\xi_1, \xi_2, \cdots, \xi_n$ is a maximally linear independent solution set of homogeneous linear equation (2).

Demonstrate, First prove that ξ_1, ξ_2, $\cdots$, ξ_n is linearly independent. With a set of numbers $y_1, y_2, \cdots, y_{n-r}$

$$y_1\xi_1 + y_2\xi_2 + \cdots y_{n-r}\xi_{n-r} = 0$$

or

$$\begin{cases} -k_{1\,r+1}y_1 - k_{1\,r+2}y_2 - \cdots - k_{1\,n}y_{n-r} = 0 \\ -k_{2\,r+1}y_1 - k_{2\,r+2}y_2 - \cdots - k_{2\,n}y_{n-r} = 0 \\ \qquad \cdots \\ -k_{r\,r+1}y_1 - k_{r\,r+2}y_2 - \cdots - k_{r\,n}y_{n-r} = 0 \\ 1y_1 + 0y_2 + \cdots + 0y_{n-r} = 0 \\ 0y_1 + 1y_2 + \cdots + 0y_{n-r} = 0 \\ \qquad \cdots \\ 0y_1 + 0y_2 + \cdots + 1y_{n-r} = 0 \end{cases}$$

The solution: $y_1 = y_2 = \cdots y_{n-r} = 0$.So $\xi_1, \xi_2, \cdots, \xi_n$ are linearly independent.From equation(7)we know that,any one of the solution of equation set(2)is one of the Linear combinations of $\xi_1, \xi_2, \cdots, \xi_n$.The theorem is proved.

The following demonstrates that any solution of the linear equations (2) can be linearly expressed by $\xi_1, \xi_2, \cdots, \xi_n$.

Let $b = 0$, Get the derived set(2) of equations (1), then apply elementary transformation on its augmented matrix $(A \vdots 0)$,the in(3) $d_1 = d_2 = \cdots = d_r = 0$, or $\eta = 0$. Under this circumstance,(7)is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = c_1\xi_1 + c_2\xi_2 + \cdots + c_{n-r}\xi_{n-r} \tag{8}$$

Demonstration finished.

Definition 1: A maximal independent set of solution set of a homogeneous linear equations is called a basic solution system of the equations.

Example 1: What is the basic solution series of homogeneous linear equations $\begin{cases} x_1+2x_2+x_3-2x_4=0 \\ 2x_1+3x_2-x_4=0 \\ x_1-x_2-5x_3+7x_4=0 \end{cases}$?

Solution: The following is the corresponding coefficient matrix and transform it into the simplest form of rows using elementary transformation

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 0 & -1 \\ 1 & -1 & -5 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

That is, the original equations have the same solution with the following equations

$$\begin{cases} x_1 = 3x_3 - 4x_4 \\ x_2 = -2x_3 + 3x_4 \end{cases}$$

Let free variables $x_3 = c_1$, $x_4 = c_2$,(c_1, c_2 are arbitrary constants)then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3c_1 - 4c_2 \\ -2c_1 + 3c_2 \\ 1c_1 + 0c_2 \\ 0c_1 + 1c_2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} = c_1\xi_1 + c_2\xi_2$$

From the theorem 1 and definitions 1, ξ_1, ξ_2 are a basic solution system of the original equations.

Example 2: Use the basic solution system to represent all the solutions of the following linear equations.

$$\begin{cases} x_1+2x_2+x_3-2x_4=3 \\ 2x_1+3x_2-x_4=5 \\ x_1-x_2-5x_3+7x_4=0 \end{cases}$$

Solution The following is the corresponding augmentation matrix and transform it to the simplest form of rows using elementary transformation

$$\begin{bmatrix} 1 & 2 & 1 & -2 & 3 \\ 2 & 3 & 0 & -1 & 5 \\ 1 & -1 & -5 & 7 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4 & 1 \\ 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That is, the original system of equations has the same solution with $\begin{cases} x_1 = 3x_3 - 4x_4 + 1 \\ x_2 = -2x_3 + 3x_4 + 1 \end{cases}$,

Let $x_3 = c_1$, $x_4 = c_2$, where c_1, c_2 are arbitrary constants, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3c_1 - 4c_2 + 1 \\ -2c_1 + 3c_2 + 1 \\ 1c_1 + 0c_2 \\ 0c_1 + 1c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \eta + c_1\xi_1 + c_2\xi_2 .$$

3. Conclusions

This paper presents a new method for making out the basic solution system of homogeneous linear equations:

(1) The coefficient matrix A of the linear equations is transformed into a row minimal matrix, and an equivalent equation system is written according to the row minimal matrix,

(2) Let free variables be arbitrary constants $c_i(i = 1,2 \cdots n - r)$,

(3) Propose a common factor (arbitrary constant) $c_i(i = 1,2 \cdots n - r)$ using the nature of the vector itself,inferred :

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & x_{r+2} & \cdots & x_n \end{bmatrix}^T = c_1\xi_1 + c_2\xi_2 + \cdots + c_{n-r}\xi_{n-r} ,$$

Where ξ_1 , ξ_2 , $\cdots$, ξ_n is a basic solution system of the corresponding homogeneous linear equations.

This solution method is applicable to both homogeneous and non-homogeneous linear equations. The solution to these two equations is completely the same and easy to use. The whole process only needs to simplify the general solution of the system of equations, which makes it easier for students to understand the nature of the basic solution system, and it is also convenient for them to establish the connection between the previous knowledge points.

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