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The Spectral Properties of the Main Operator of a Kind of Pumping Well Production System with Early Warning Function

Yuhong Cui[1,a,*], Youde Tao[1,b]

[1]*College of Mathematics and Statistics, Xinyang Normal University, Xinyang, Henan, China*
[a]*cruby_ok@126.com*, [b]*xynutyd@xynu.edu.cn*
[*]*Corresponding author*

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Abstract: Using the conjugate space theory of functional analysis, the eigenvalue problem of the main operator of the pumping well production system with early warning function was studied, and the spectral point distribution of the main operator of the system was described.

1. Introduction

Pumping well production system with early warning function refers to a kind of complex system specially used for crude oil production, which can accurately find the abnormal situation of oil well production and control it in advance according to the early warning signal, or continue production or maintenance, and belongs to the category of repairable system in reliability mathematics. To study the reliability of such systems, the primary task is to study the reliability indexes such as transient reliability and steady-state availability. These reliability indexes are closely related to the time-dependent solution of the system, the steady-state solution of the system, and the distribution law of the spectral points of the main operators of the system [1−10]. Among them, Geni Gupur studied the spectral characteristics of a system composed of two identical components and a repair device by using the theory of linear operator semigroup, and obtained the asymptotic properties of the time-dependent solution of the system [2]; Guo Weihua et al. discussed the eigenvalue problem of the main operator of two parallel repairable systems with different components by using the theory of functional analysis, and proved the stability of the system solution[3]; Xu Houbao et al.[4] studied the spectral point distribution of software regeneration system by using C_0-semigroup theory, and proved the existence and uniqueness of the non-negative steady-state solution of the system. On the basis of the above literature, this paper will apply functional analysis and linear operator semigroup theory to study the geometric multiplicity and algebraic multiplicity of eigenvalues corresponding to the main operator and its conjugate operator of a kind of pumping well oil production system with early warning function, and describe the spectral point distribution law of the main operator of the system.

2. Mathematical Model

Pumping well production system refers to a kind of complex system specially used for crude oil production, which usually consists of various components, such as pumping unit, oil pump, sucker rod string, motor, reducer, early warning device and repairman. For the convenience of discussion, this paper assumes that the pumping system with early warning function is composed of back pressure, casing pressure, early warning device and a maintainer. When oil wells are in production, wellhead back pressure and casing pressure have warning function. When the back pressure and casing pressure are abnormal, exceeding or lower than the preset safety production parameters of the system, the system warning device gives an alarm, but the maintenance personnel ignore the alarm and the oil well continues to produce; When back pressure and casing pressure fail, causing pipeline blockage or oil well shutdown, the oil well stops production, and the maintenance personnel will repair the failed parts, the component and the system can be repaired intact, make them work properly. Therefore, the immediate state of the oil production system of the repairable pumping unit well can be subdivided into the following situations, and the state transition diagram of the corresponding system can be shown by Figure 1.

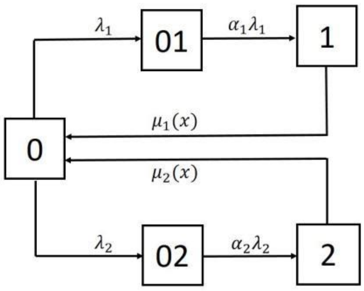

Figure 1: State transition diagram of the System

(1) State 0 indicates that the back pressure and casing pressure of the oil well are normal, and the system is in a normal working state;

(2) State 01 indicates that the back pressure of the oil well is abnormal, and the system gives an early warning prompt, but it will not be repaired for the time being and continue to work state;

(3) State 1 indicates that the back pressure of the oil well has failed, and the system can no longer work, it will be repaired state;

(4) State 02 indicates that the casing pressure of the oil well is abnormal, and the system gives an early warning prompt, but it will not be repaired for the time being and continue to work state;

(5) State 2 indicates that the casing pressure of the oil well has failed, and the system can no longer work, it will be repaired state;

In order to facilitate the establishment of model and model analysis, the following general assumptions can be made according to the state transition diagram of pumping well production system with early warning function:

(1) The system has three states: normal state, early warning state and fault state.

(2) The failure rate of oil well back pressure and casing pressure is constant, and the repair rate of back pressure and casing pressure is non-constant.

(3) All kinds of faults are independent of each other in statistical sense.

(4) The normal working time of oil well back pressure and casing pressure system obeys exponential distribution function $F = 1 - e^{-\lambda t}$, $t \geq 0$, $\lambda > 0$.

(5) The repair time of oil well back pressure and casing pressure system obeys the general distribution function $G = 1 - e^{-\mu_i(x)t}, t \geq 0, \mu_i(x) > 0, i = 1, 2$

(6) After oil well back pressure, casing pressure and system repair, it works normally.

At this time, the supplementary variable $X_i(t)$ is introduced by using the methods in literatures[2-4, 8,10], it represents the maintenance time that the component i fails and has been spent at t , that is, the maintenance time of oil well back pressure or casing pressure from the beginning of maintenance until now, $i = 1,2$, then the mathematical model[13] of the recoverable pumping well production system with early warning function can be described as

$$
\begin{cases}
\dfrac{dP_0(t)}{dt} = -(\lambda_1+\lambda_2)P_0(t) + \sum_{i=1}^{2}\int_0^\infty \mu_i P_i(x,t)dx, \\
\dfrac{dP_{01}(t)}{dt} = -\alpha_1\lambda_1 P_{01}(t) + \lambda_1 P_0(t), \\
\dfrac{dP_{02}(t)}{dt} = -\alpha_2\lambda_2 P_{02}(t) + \lambda_2 P_0(t), \\
\dfrac{\partial P_i(x,t)}{\partial x} + \dfrac{\partial P_i(x,t)}{\partial t} = -\mu_i(x)P_i(x,t),\ i=1,2, \\
P_1(0,t) = \alpha_1\lambda_1 P_{01}(t), P_2(0,t) = \alpha_2\lambda_2 P_{02}(t), \\
P_0(0) = 1, P_{01}(0) = P_{02}(0) = P_1(x,0) = P_2(x,0) = 0.
\end{cases} \tag{1}
$$

Where $(x,t) \in [0,\infty)\times[0,\infty); P_0(t)$ means the probability that the system is in the state of 0 at time t; $P_{0i}(t)$ represents the probability that the system will be in the state of $0i$ (i = 1,2) at time t; $P_i(x,t)$ means the probability that the system is in the state of i(i = 1,2) at time t and the dwell time (i.e. the repair time of back pressure or casing pressure) is x (i.e. the probability of system failure caused by back pressure failure or casing pressure failure within [x,x + dx]). In addition, λ_1 indicates the failure rate of back pressure, λ_2 indicates the failure rate of casing pressure, and α_i indicates the sensitivity coefficient of the system in state i(i = 1,2); Here $\alpha_i\lambda_i$ represents the sensitive failure rate of the system in state i(i = 1,2), and $\mu_i(x)$ indicates the repair rate of the system residing in state i(i = 1,2) for time x, and meets the requirements

$$
\begin{aligned}
&0 \le \mu_i(x) < \infty, M = \sup_{x\in[0,\infty)} \mu_i(x), \int_0^\infty \mu_i(x)dx = \infty, \\
&0 < \lim_{x\to\infty}\frac{1}{x}\int_0^x \mu_i(\tau)d\tau = \mu_i < \infty, (i=1,2).
\end{aligned} \tag{2}
$$

As the mathematical model of pumping well production system with early warning function contains both integral and differential, it is difficult to deal with it directly. Therefore, it is necessary to transform it into an abstract Cauchy problem in Banach space before reliability analysis. For this reason, referring to the practices of references [2,8], the state space X is selected as

$$
X = \left\{ P \in \mathbb{R}^3 \times (L^1[0,+\infty))^2 \,\middle|\, \|P\| = |P_0| + \sum_{i=1}^{2}|P_{0i}| + \sum_{i=1}^{2}\|P_i(x)\|_{L^1[0,+\infty)} < \infty \right\}, \tag{3}
$$

Where $P = (P_0, P_{01}, P_{02}, P_1(x), P_2(x)) \in X$, at this time, you can verify that $(X, \|\cdot\|)$ is a Banach space. In addition, the definition operators A, B and their definition fields $D(A)$, $D(B)$ are

$$
AP = diag\Big(-\lambda_1-\lambda_2, -\alpha_1\lambda_1, -\alpha_2\lambda_2, -\big(\frac{d}{dx}+\mu_1(x)\big), -\big(\frac{d}{dx}+\mu_2(x)\big)\Big)P, \tag{4}
$$

$$
D(A) = \left\{ P \in X \,\middle|\, \begin{array}{l} \dfrac{d}{dx}P_i(x) + \mu_i(x)P_i(x) \in L^1[0,+\infty), P_i(x)(i=1,2) \text{is an absolutely continuous function} \\ \text{and satisfies } P(0) = \big(P_0, P_{01}, P_{02}, P_1(0), P_2(0)\big) = \big(P_0, P_{01}, P_{02}, \alpha_1\lambda_1 P_{01}, \alpha_2\lambda_2 P_{02}\big) \end{array} \right\}, \tag{5}
$$

$$BP=\begin{pmatrix} 0 & 0 & 0 & \int_0^\infty \mu_1(x)\mathrm{d}x & \int_0^\infty \mu_2(x)\mathrm{d}x \\ \lambda_1 & 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_{01} \\ P_{02} \\ P_1(x) \\ P_2(x) \end{pmatrix}, D(B)=X. \tag{6}$$

Therefore, the system (1) can be expressed as an abstract Cauchy problem on Banach space X :

$$\begin{cases} \dfrac{\mathrm{d}P(t)}{\mathrm{d}t}=(A+B)P(t), t\geq 0, \\ P(0)=(1,0,0,0,0)^T, \\ P(t)=(P_0(t),P_{01}(t),P_{02}(t),P_1(x,t),P_2(x,t))^T. \end{cases} \tag{7}$$

3. Spectral Properties of the Main Operator of the System

In order to study the spectral properties of the main operator of the system, the definition of X conjugate space is given here. According to the reference [11], the conjugate space of X can be defined as

$$X^*=\left\{q^*\in R^3\times\left(L^\infty[0,\infty]\right)^2 \,\middle|\, \|q^*\|=\max\left\{|q_0^*|,|q_{01}^*|,|q_{02}^*|,\|q_1^*\|_{L^\infty[0,\infty)},\|q_2^*\|_{L^\infty[0,\infty)}\right\}\right\}, \tag{8}$$

And it is easy to verify that $(X,\|\cdot\|)$ is a Banach space.

Theorem 3.1. The conjugate operator of $(A+B)$ is $(A+B)^*$, and for any $q^*\in D(G)$, there is always

$$(A+B)^*q^*=(G+F)q^*, \tag{9}$$

Among

$$Gq^*(x)=\begin{pmatrix} -\lambda_1-\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mu_1(x) & 0 & 0 & \dfrac{\mathrm{d}}{\mathrm{d}x}-\mu_1(x) & 0 \\ \mu_2(x) & 0 & 0 & 0 & \dfrac{\mathrm{d}}{\mathrm{d}x}-\mu_2(x) \end{pmatrix}\begin{pmatrix} q_0^* \\ q_{01}^* \\ q_{02}^* \\ q_1^*(x) \\ q_2^*(x) \end{pmatrix}. \tag{10}$$

$$Fq^*(x)=\begin{pmatrix}0&0&0&0&0\\0&0&0&\alpha_1\lambda_1&0\\0&0&0&0&\alpha_2\lambda_2\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}q_0^*\\q_{01}^*\\q_{02}^*\\q_1^*(0)\\q_2^*(0)\end{pmatrix}+\begin{pmatrix}0&\lambda_1&\lambda_2&0&0\\0&-\alpha_1\lambda_1&0&0&0\\0&0&-\alpha_2\lambda_2&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}q_0^*\\q_{01}^*\\q_{02}^*\\q_1^*(x)\\q_2^*(x)\end{pmatrix} \tag{11}$$

$$D(G)=\left\{q^*\in X^*\left|\begin{array}{l}\dfrac{\mathrm{d}q^*(x)}{\mathrm{d}x}\text{ exists, and } q_1^*(\infty)=q_2^*(\infty)=a,\\ \text{and } a \text{ is a constant which is unrelated to } q_1^*(x),q_2^*(x).\end{array}\right.\right\}. \tag{12}$$

Proof. For any $P\in D(A)$, $q*\in D(G)$, from the partial integration formula and the boundary conditions, we get

$$\begin{aligned}
&\langle (A+B)P,q^*\rangle\\
&=\left[-(\lambda_1+\lambda_2)P_0+\sum_{i=1}^{2}\int_0^\infty \mu_i(x)P_i(x)\mathrm{d}x\right]q_0^*\\
&\quad+\sum_{i=1}^{2}(-\alpha_i\lambda_iP_{0i}+\lambda_iP_0)q_{0i}^*+\sum_{i=1}^{2}\int_0^\infty\left[-\frac{\mathrm{d}P_i(x)}{\mathrm{d}x}-\mu_i(x)P_i(x)\right]q_i^*(x)\mathrm{d}x\\
&=-(\lambda_1+\lambda_2)P_0q_0^*+\sum_{i=1}^{2}\int_0^\infty \mu_i(x)q_0^*P_i(x)\mathrm{d}x+\sum_{i=1}^{2}(-\alpha_i\lambda_iq_{0i}^*P_{0i}+\lambda_iq_{0i}^*P_0)-\int_0^\infty\frac{\mathrm{d}P_1(x)}{\mathrm{d}x}q_1^*(x)\mathrm{d}x\\
&\quad-\int_0^\infty\mu_1(x)P_1(x)q_1^*(x)\mathrm{d}x-\int_0^\infty\frac{\mathrm{d}P_2(x)}{\mathrm{d}x}q_2^*(x)\mathrm{d}x-\int_0^\infty\mu_2(x)P_2(x)q_2^*(x)\mathrm{d}x\\
&=-(\lambda_1+\lambda_2)q_0^*P_0+\sum_{i=1}^{2}\int_0^\infty \mu_i(x)q_0^*P_i(x)\mathrm{d}x+\sum_{i=1}^{2}(-\alpha_i\lambda_iq_{0i}^*P_{0i}+\lambda_iq_{0i}^*P_0)\\
&\quad-[q_1^*(x)P_1(x)]\Big|_0^\infty+\int_0^\infty\frac{\mathrm{d}q_1^*(x)}{\mathrm{d}x}P_1(x)\mathrm{d}x-\int_0^\infty\mu_1(x)q_1^*(x)P_1(x)\mathrm{d}x\\
&\quad-[q_2^*(x)P_2(x)]\Big|_0^\infty+\int_0^\infty\frac{\mathrm{d}q_2^*(x)}{\mathrm{d}x}P_2(x)\mathrm{d}x-\int_0^\infty\mu_2(x)q_2^*(x)P_2(x)\mathrm{d}x
\end{aligned} \tag{13}$$

$$\begin{aligned}
&=-(\lambda_1+\lambda_2)q_0^*P_0+\sum_{i=1}^{2}\int_0^\infty \mu_i(x)q_0^*P_i(x)\mathrm{d}x+\sum_{i=1}^{2}(-\alpha_i\lambda_iq_{0i}^*P_{0i}+\lambda_iq_{0i}^*P_0)\\
&\quad+q_1^*(0)P_1(0)+\int_0^\infty\frac{\mathrm{d}q_1^*(x)}{\mathrm{d}x}P_1(x)\mathrm{d}x-\int_0^\infty\mu_1(x)q_1^*(x)P_1(x)\mathrm{d}x\\
&\quad+q_2^*(0)P_2(0)+\int_0^\infty\frac{\mathrm{d}q_2^*(x)}{\mathrm{d}x}P_2(x)\mathrm{d}x-\int_0^\infty\mu_2(x)q_2^*(x)P_2(x)\mathrm{d}x\\
&=-(\lambda_1+\lambda_2)q_0^*P_0+\sum_{i=1}^{2}\int_0^\infty \mu_i(x)q_0^*P_i(x)\mathrm{d}x+\sum_{i=1}^{2}(-\alpha_i\lambda_iq_{0i}^*P_{0i}+\lambda_iq_{0i}^*P_0)\\
&\quad+\alpha_1\lambda_1q_1^*(0)P_{01}+\int_0^\infty\frac{\mathrm{d}q_1^*(x)}{\mathrm{d}x}P_1(x)\mathrm{d}x-\int_0^\infty\mu_1(x)q_1^*(x)P_1(x)\mathrm{d}x\\
&\quad+\alpha_2\lambda_2q_2^*(0)P_{02}+\int_0^\infty\frac{\mathrm{d}q_2^*(x)}{\mathrm{d}x}P_2(x)\mathrm{d}x-\int_0^\infty\mu_2(x)q_2^*(x)P_2(x)\mathrm{d}x
\end{aligned}$$

$$=-(\lambda_1+\lambda_2)q_0^*P_0+\sum_{i=1}^{2}\int_0^\infty\Big[\frac{\mathrm{d}q_i^*(x)}{\mathrm{d}x}-\mu_i(x)q_i^*(x)+\mu_i(x)q_0^*\Big]P_i(x)\mathrm{d}x$$
$$+(\lambda_1q_{01}^*+\lambda_2q_{02}^*)P_0+\big[-\alpha_1\lambda_1q_{01}^*+\alpha_1\lambda_1q_1^*(0)\big]P_{01}+\big[-\alpha_2\lambda_2q_{02}^*+\alpha_2\lambda_2q_2^*(0)\big]P_{02}$$
$$=\int_0^\infty\Big[\frac{\mathrm{d}q_1^*(x)}{\mathrm{d}x}-\mu_1(x)q_1^*(x)+\mu_1(x)q_0^*\Big]P_1(x)\mathrm{d}x+\int_0^\infty\Big[\frac{\mathrm{d}q_2^*(x)}{\mathrm{d}x}-\mu_2(x)q_2^*(x)+\mu_2(x)q_0^*\Big]P_2(x)\mathrm{d}x$$
$$-(\lambda_1+\lambda_2)q_0^*P_0+(\lambda_1q_{01}^*+\lambda_2q_{02}^*)P_0+[-\alpha_1\lambda_1q_{01}^*+\alpha_1\lambda_1q_1^*(0)]P_{01}+[-\alpha_2\lambda_2q_{02}^*+\alpha_2\lambda_2q_2^*(0)]P_{02}$$
$$=\langle P,(G+F)q^*\rangle.$$

Thus, from the definition of conjugate operator, the conclusion of Theorem 3.1 holds.
Theorem 3.2. 0 is the eigenvalue of A + B with geometric multiplicity of 1.
Proof. Discuss the equation (A + B) P = 0, that is

$$\begin{cases}-(\lambda_1+\lambda_2)P_0+\sum_{i=1}^{2}\int_0^\infty\mu_i(x)P_i(x)\mathrm{d}x=0,\\ -\alpha_1\lambda_1P_{01}+\lambda_1P_0=0,\\ -\alpha_2\lambda_2P_{02}+\lambda_2P_0=0,\\ \frac{\mathrm{d}P_i(x)}{\mathrm{d}x}+\mu_i(x)P_i(x)=0, i=1,2,\\ P_1(0)=\alpha_1\lambda_1P_{01}, P_2(0)=\alpha_2\lambda_2P_{02}.\end{cases} \tag{14}$$

Solve the above equations

$$\begin{cases}P_i(x)=P_i(0)e^{-\int_0^x\mu_i(\tau)\mathrm{d}\tau}, i=1,2,\\ -(\lambda_1+\lambda_2)P_0+\sum_{i=1}^{2}\int_0^\infty\mu_i(x)P_i(x)\mathrm{d}x=0.\end{cases} \tag{15}$$

Let $P_0>0$, then the equations (14), (15) have

$$\begin{cases}P_{01}=\frac{1}{\alpha_1}P_0,\\ P_{02}=\frac{1}{\alpha_2}P_0,\\ P_1(x)=\lambda_1e^{-\int_0^x\mu_1(\tau)\mathrm{d}\tau}P_0,\\ P_2(x)=\lambda_2e^{-\int_0^x\mu_2(\tau)\mathrm{d}\tau}P_0.\end{cases} \tag{16}$$

So from the equations (16) and Fubini theorem, there are

$$
\begin{aligned}
\| P\| &= |P_0| + \sum_{i=1}^{2} |P_{0i}| + \sum_{i=1}^{2} \|P_i(x)\|_{L^1[0,+\infty)} \\
&\le |P_0| + \frac{1}{\alpha_1}|P_0| + \frac{1}{\alpha_2}|P_0| + \lambda_1 |P_0| + \lambda_2 |P_0| \\
&= (1 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \lambda_1 + \lambda_2)|P_0| \\
&< \infty.
\end{aligned} \tag{17}
$$

Therefore $P_i(x) \in L^1[0,\infty)$, i = 1,2, thus $P = (P_0, P_{01}, P_{02}, P_1(x), P_2(x))$ is the eigenvector of the 0 eigenvalue corresponding to the operator A + B.

Take any Q = (1,1,1,1,1), then

$$
\langle P, Q\rangle = P_0 + \sum_{i=1}^{2} P_{0i} + \sum_{i=1}^{2} \int_0^\infty P_i(x)\mathrm{d}x > 0, \tag{18}
$$

That is, for any P∈D(A+B), there is always ⟨(A+B) P, Q⟩ = 0, which indicates that 0 is the eigenvalue of A + B with geometric multiplicity of 1.

Theorem 3.3. 0 is the eigenvalue of conjugate operator $(A+B)^*$ whose geometric multiplicity is 1, and the eigenvector relative to 0 is positive.

Proof. Discuss the equation $(A+B)^* q^* = (G+F)q^* = 0$, and from the Theorem 3.1, there is

$$
-(\lambda_1 + \lambda_2)q_0^* + \lambda_1 q_{01}^* + \lambda_2 q_{02}^* = 0,, \tag{19}
$$

$$
-\alpha_1 \lambda_1 q_{01}^* + \alpha_1 \lambda_1 q_1^*(0) = 0, \tag{20}
$$

$$
-\alpha_2 \lambda_2 q_{02}^* + \alpha_2 \lambda_2 q_2^*(0) = 0, \tag{21}
$$

$$
\frac{\mathrm{d}q_1^*(x)}{\mathrm{d}x} - \mu_1(x)q_1^*(x) + \mu_1(x)q_0^* = 0, \tag{22}
$$

$$
\frac{\mathrm{d}q_2^*(x)}{\mathrm{d}x} - \mu_2(x)q_2^*(x) + \mu_2(x)q_0^* = 0, \tag{23}
$$

$$
q_1^*(\infty) = q_2^*(\infty) = a. \tag{24}
$$

By solving the equations

$$
(\lambda_1 + \lambda_2)q_0^* = \lambda_1 q_{01}^* + \lambda_2 q_{02}^*, \tag{25}
$$

$$
q_1^*(0) = q_{01}^*,\ q_2^*(0) = q_{02}^*, \tag{26}
$$

$$
q_1^*(x) = q_1^*(0)e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} - e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \int_0^x \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau, \tag{27}
$$

$$
q_2^*(x) = q_2^*(0)e^{\int_0^x \mu_2(\xi)\mathrm{d}\xi} - e^{\int_0^x \mu_2(\xi)\mathrm{d}\xi} \int_0^x \mu_2(\tau) q_0^* e^{-\int_0^\tau \mu_2(\xi)\mathrm{d}\xi} \mathrm{d}\tau. \tag{28}
$$

We multiply both ends of (27), (28) by $e^{-\int_0^x \mu_1(\xi)\mathrm{d}\xi}, e^{-\int_0^x \mu_2(\xi)\mathrm{d}\xi}$, and make x → ∞ and by using (24), we can get

$$q_1^*(0) = \int_0^\infty \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau, \tag{29}$$

$$q_2^*(0) = \int_0^\infty \mu_2(\tau) q_0^* e^{-\int_0^\tau \mu_2(\xi)\mathrm{d}\xi} \mathrm{d}\tau. \tag{30}$$

Then substituting (29) into (27) to get

$$\begin{aligned} q_1^*(x) &= e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \int_0^\infty \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau - e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \int_0^x \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau \\ &= e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \Big[\int_0^\infty \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau - \int_0^x \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau\Big] \\ &= e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \int_x^\infty \mu_1(\tau) q_0^* e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \mathrm{d}\tau \\ &= -q_0^* e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \int_x^\infty \frac{\mathrm{d}e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi}}{\mathrm{d}\tau} \\ &= -q_0^* e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \Big[e^{-\int_0^\tau \mu_1(\xi)\mathrm{d}\xi} \Big|_x^\infty\Big] \\ &= -q_0^* e^{\int_0^x \mu_1(\xi)\mathrm{d}\xi} \Big[0 - e^{-\int_0^x \mu_1(\xi)\mathrm{d}\xi}\Big] \\ &= q_0^*. \end{aligned} \tag{31}$$

Similarly, substituting (30) into (28) to get

$$q_2^*(x) = q_0^* \tag{32}$$

From (26) know $q_1^*(0) = q_{01}^* = q_0^*,\ q_2^*(0) = q_{02}^* = q_0^*$, to make the corresponding $q^* \in X^*$, there must be $q_i^*(x) \in L^\infty_{[0,\infty)}, i = 1, 2$. Note that the function $e^{\mu_i x}$ is unbounded, so $q_i^*(x) \in L^\infty_{[0,\infty)}$ if and only if $q_i^*(0) = q_0^*$, then $q_i^*(x) = q_0^*, i = 1, 2$, and then $q^* = (q_0^*, q_{01}^*, q_{02}^*, q_1^*(x), q_2^*(x))$. At this point, combining (31),(32), we have

$$|||q^*||| = \max\{|q_0^*|, |q_{01}^*|, |q_{02}^*|, \|q_1^*\|_{L^1[0,+\infty)}, \|q_2^*\|_{L^1[0,+\infty)}\} = |q_0^*| < \infty. \tag{33}$$

That is $q^* \in X^*$. It indicates that 0 is the eigenvalue of $(A+B)^*$, In addition, it is easy to seefrom (31), (32) the eigenvectors $(q_0^*, q_{01}^*, q_{02}^*, q_1^*(x), q_2^*(x)) = (q_0^*, q_0^*, q_0^*, q_0^*, q_0^*)$ corresponding to the eigen-value 0 generates one-dimensional linear space, that is, the geometric multiplicity of 0 is 1.

Theorem 3.4.0 is the eigenvalue of operator A + B in X with algebraic multiplicity of 1.

Proof. From the Theorem 3.2, the eigenvector corresponding to the eigenvalue 0 in X can be expressed as

$$P = a\left(P_0, \frac{1}{\alpha_1}P_0, \frac{1}{\alpha_2}P_0, \lambda_1 e^{-\int_0^x \mu_1(\tau)\mathrm{d}\tau}P_0, \lambda_2 e^{-\int_0^x \mu_2(\tau)\mathrm{d}\tau}P_0\right),\ a \neq 0,\ P_0 \neq 0,\ a \in R. \tag{34}$$

From the Theorem 3.3, the eigenvector corresponding to the eigenvalue of 0 in X^* is

$$q^* = b\left(q_0^*, q_0^*, q_0^*, q_0^*, q_0^*\right),\ b \neq 0,\ q_0^* \neq 0,\ b \in R. \tag{35}$$

Therefore

$$\langle P, q^* \rangle = abP_0 q_0^*\left(1, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \lambda_1 e^{-\int_0^x \mu_1(\tau)\mathrm{d}\tau}, \lambda_2 e^{-\int_0^x \mu_2(\tau)\mathrm{d}\tau}\right) \neq 0. \tag{36}$$

This indicates that the algebraic index of 0 is 1, and since the geometric multiplicity of 0 is 1, the algebraic multiplicity of 0 is 1.

By synthesizing the above theorems, the main result of this paper is obtained: the spectral properties of the system operator A + B. For this reason, there are the following theorems:

Theorem 3.5.[12] $\{\gamma \in \mathbb{C} \mid \mathrm{Re}\gamma > 0 \text{ or } \gamma = ai, a \in \mathbb{R}, a \neq 0\} \subset \rho(A+B)$, That is, all points on the imaginary axis except 0 belong to the resolvent set of the operator A + B, 0 is the eigenvalue of (A + B) and $(A+B)^*$ with geometric multiplicity of 1, 0 is the eigenvalue of A + B with algebraic multiplicity of 1, and the eigenvectors corresponding to 0 are positive. In addition, 0 is a strictly dominant eigenvalue of A + B.

4. Conclusions

In this paper, by using the conjugate space theory of functional analysis[14], the algebraic multiplicity and geometric multiplicity of 0 eigenvector of the system operator A + B and its conjugate operator $(A+B)^*$ are discussed, and the spectral properties of A + B are characterized. The results provide a solid theoretical basis for further study of the asymptotic stability of the solution of the repairable system [15].

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