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A New Error Estimation Method of the Quadratic Virtual Element Method for Obstacle Problems

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Abstract: This paper presents a novel approach to the error estimation of the quadratic virtual element method for unilateral obstacle problems. This method is concise and has the potential for generalization. This paper also gives numerical experimental results on convex and non-convex polygons to verify the theoretical results.

1. Introduction

Obstacle problems are important and fundamental problems in variational inequality theory. The core of this kind of problems lies in finding a function that lies above (or below) the obstacle, as well as minimizing an energy functional. These problems have wide applications in contact problems, phase-transition problems, seepage problems, and option pricing problems, etc.

Let Ω be a bounded open set in R^2 with Lipschitz boundary $\partial\Omega$. Let $f(x)\in L^2(\Omega)$, $\psi(x)\in H^1(\Omega)$, and $\psi\leq 0$ on the boundary $\partial\Omega$. The unilateral obstacle problem can be formulated as

$$\begin{cases} \text{Find } u(x)\in K, \text{ such that} \\ a(u,v-u)\geq(f,v-u), \ \forall\ v\in K. \end{cases} \tag{1}$$

where $K=\{v\in H_0^1(\Omega): v\geq\psi, \text{ a.e. in } \Omega\}$ is a closed convex set in the space $H_0^1(\Omega)$ denoting the collection of admissible displacement functions and $a(u,v)=\int_\Omega \nabla u\nabla v\mathrm{d}x$, $(f,v)=\int_\Omega fv\mathrm{d}x$. This is a representative first class of second-order variational inequalities. The bilinear form $a(\cdot,\cdot)$ satisfies continuity and coercion on the space $H_0^1(\Omega)$, namely there exist positive constants M,α such that

$$a(u,v)\leq M|u|_1|v|_1,\ a(v,v)\geq\alpha|v|_1^2,\ \forall\ u,v\in H_0^1(\Omega).$$

By Georges Duvaut et al. [1], the problem (1) has a unique solution. According to the article of Weimin Han et al. [2], if $u\in H^2(\Omega)$, the differential form that is equivalent to (1) is

$$-\Delta u-f\geq 0,\ u-\psi\geq 0,\ (-\Delta u-f)(u-\psi)=0,\ a.e. \text{ in } \Omega. \tag{2}$$

So, the problem (1) has a unique solution[1]. According to Weimin Han[2], if $u\in H^2(\Omega)$, the differential form that is equivalent to (1) is

$$\begin{cases}-\Delta u\geq f, \text{ in } \Omega^0,\\ -\Delta u=f, \text{ in } \Omega^+=\Omega\backslash\Omega^0,\end{cases} \tag{3}$$

where Ω^0 is the area that the elastic membrane contacts with the obstacle, and Ω^+ is the non-contact area.

David Kinderlehrer etal[4] have proved the following result about the regularity of solution the obstacle problem: if $\partial\Omega$ is smooth, $f\in L^\infty(\Omega)\cap BV(\Omega)$ and $\psi\in C^3(\bar{\Omega})$, then the solution of the problem (1) has the regularity $u\in W^{s,p}(\Omega)$ with any $1<p<\infty$ and $s<2+1/p$. Here, $BV(\Omega)$ is the space of bounded variation functions[5].

Actually, the classical finite element method (FEM) and discontinuous Galerkin (DG) methods have already been used to solve the obstacle problem respectively by Lieheng Wang and Fei Wang et al[6,7]. The virtual element method (VEM) was first proposed by L. Beirão da Veiga for elliptic problems as a generalization of the classical FEM[8]. Compared to FEM, VEM has achieved significant advancements in the geometric shapes and convexity of the elements. In FEM, common element shapes include convex shapes such as triangles, rectangles, tetrahedrons and cuboids, etc. However, the shapes of VEM elements can be any polygon or polyhedron with no requirement for the convexity of the elements[9] . This advancement has brought convenience in aspects such as grid refinement [10], complex domain pasting and suspended nodes processing[11]. Due to its unique advantage, the VEM has also attracted the attention of scholars working on variational inequality problems. Fei Wang et al uses the VEM to solve a simplified friction problem and derives the optimal error estimate in H^1-norm for the lowest-order VEM[12]. Fang Feng et al develope a virtual element methods for solving elliptic variational inequalities (EVIs) of the second kind[13]. Fei Wang et al establishes a priori error estimates of VEMs for the obstacle problem[14]. The authors prove that the lowest-order VEM achieves the optimal convergence order, and suboptimal order is obtained for the quadratic VEM. Jiali Qiu et al explores the analysis and numerical solution of a fourth-order history-dependent hemivariational inequality, and a priori error estimates for the fully discrete scheme are established[15].

As a pioneering achievement, Fei Wang et al first analyzed errors in solving obstacle problems by using lowest-order and high-order virtual elements, and has obtained optimal-order and suboptimal-order error estimates, respectively[14]. In Theorem 3.4 of the paper, an error estimate of quadratic virtual element for obstacle problems was derived. The key technique involved dividing all elements into three groups: the elements in Ω^0, the elements in Ω^+, and others, then estimating errors in each group. The proof process is lengthy and complicated.

In this article, we present a relatively concise proof. We divide the error into three parts R_1,R_2 and R_3, and the first two parts can be estimated quite easily. To estimate R_3, we divide each element T into two parts T^0 and T^+, where $T^0=T\cap\Omega^0$ and $T^+=T\cap\Omega^+$. Using the property (3) on Ω^+ and the interpolation theorem, we can get the estimate of R_3 in a natural and concise manner. This idea can also be applied to the analysis of other variational inequality problems.

The rest of the paper is organized as follows. In Section 2 we introduce VEM and the discrete formula of the obstacle problem (1). In Section 3 we give a new H^1 semi-norm error estimation of the quadratic VEM for the problem (1). Finally, in Section 4 we show the numerical results to verify the theoretical analysis.

2. Virtual element discretization

Denote $\{\mathrm{T}_h\}$ as the sequence of decompositions of Ω, E_h as the set of all edges e of the partition T_h. For each element $K\in\mathrm{T}_h$, h_K denotes the diameter of K, and h denotes the maximum diameter of all elements in T_h. For any $K\in\mathrm{T}_h$, we give the following assumptions [8]:

Assumption 2.1. For any h, the decomposition T_h consists of a finite number of simple polygons.

Assumption 2.2. For any element $T\in\mathrm{T}_h$, there exists a constant $\varrho>0$ such that

- for every edge $e\subseteq\partial T$, we have $|e|\geq\varrho h_T$,
- T is star-shaped with respect to a ball of radius $\geq\varrho h_T$.

Under the above assumptions, by the classical Scott-Dupont theorem from , we have the following basic approximation result.

Lemma 2.1. *Under* **Assumption 2.2**, *there exists a constant C associated only with T and ϱ, such that for any s, $1\leq s\leq k+1$ and any $u\in H^{s+1}(T)$, there exists a polynomial function $u_\pi\in P_k(T)$, such that*

$$||u-u_\pi||_{0,T}+h_T||u-u_\pi||_{1,T}\leq Ch_T^s|u|_{s,T}, \tag{4}$$

where $k\in\mathrm{N}$ $(k\geq1)$, and $\mathrm{P}_k(T)$ denotes the set of polynomials of order $\leq k$ on element T.

We first define the local virtual element space V_T^k as

$$V_T^k=\{v\in H^1(T); v|_e\in\mathrm{P}_k(T),\ \forall\ e\in\partial T;\ \Delta v\in\mathrm{P}_{k-2}(T)\},\forall\ T\in\mathrm{T}_h, \tag{5}$$

and define the global virtual element space as

$$V_h^k=\{v\in H_0^1(\Omega);\ v|_T\in V_T^k,\ \forall\ T\in\mathrm{T}_h\}. \tag{6}$$

For all v in V_T^k, we give degrees of freedom of v as follows:

(D1)The values of v at the $n=n(T)$ vertices $a_i(i=1,\cdots,n)$ of T,

(D2)For each $e\in\partial T$, the values of v at k-1 distinct points on e,

(D3)The moments $\int_K vp\ \mathrm{d}x,\ \forall\ p\in\mathrm{P}_{k-2}(T)$,

where $n(T)$ denotes the number of vertices of element T. Note that the values (D1)-(D2) uniquely determine the function v on the boundary ∂T.

Especially, when $k=2$, the degrees of freedom of $v_h\in V_T^2$ are the values at the vertices, the values at the midpoints of the edges and the moments $\frac{1}{|T|}\int_T v\,\mathrm{d}x$ on the element T, respectively. It follows from that the functions in V_T^2 can be uniquely determined by the above degrees of freedom.

We define the projection operator $\Pi_{T,k}^0:L^2(T)\to\mathrm{P}_k(T)$ and $\Pi_{T,k}:V_T^k\to\mathrm{P}_k(T)$ as

$$(\Pi_{T,k}^0 v,q)=(v,q),\ \forall\ q\in\mathrm{P}_k(T),\ \mathit{for\ any}\ \ v\in L^2(T), \tag{7}$$

$$\begin{cases} a^T(\Pi_{T,k}v_h,q)=a^T(v_h,q),\ \forall\ q\in\mathrm{P}_k(T), \\ P_0(\Pi_{T,k}v_h)=P_0 v_h,\ \mathit{for\ any}\ \ v_h\in V_T^k, \end{cases} \tag{8}$$

where $P_0 v := \frac{1}{n}\sum_{i=1}^n v(a_i)$ $(k=1)$ and $P_0 v := \frac{1}{|T|}\int_T v\mathrm{d}x$ $(k\geq2)$.

We define the local discrete form of the bilinear form $a(\cdot,\cdot)$ now. For any $T\in\mathrm{T}_h$, $u_h,v_h\in V_T^k$, there is

$$a_h^T(u_h,v_h) := a^T(\Pi_{T,k}u_h,\Pi_{T,k}v_h)+S^T\big((I-\Pi_{T,k})u_h,(I-\Pi_{T,k})v_h\big),$$

where $S^T(\cdot,\cdot)$ is a symmetric positive definite bilinear form that defined as

$$S^T(u_h,v_h)=\sum_{i=1}^{N_{dof}^T}\chi_i(u_h)\chi_i(v_h),\ \forall\ u_h,v_h\in V_T^k,$$

here χ_i denotes the i-th degree of freedom of the element T, and $N_{dof}^T=\dim V_T^k$. According to L. Beirão da Veiga et al. [8], $a_h^T(\cdot,\cdot)$ satisfies the following properties:

- consistency: $a_h^T(v_h,q)=a^T(v_h,q), \forall v_h\in V_T^k, \forall q\in P_k(T)$.
- stability: there exists $c_s>0, c_b>0$ such that

$$c_s a^T(v_h,v_h)\le a_h^T(v_h,v_h)\le c_b a^T(v_h,v_h), \forall v_h\in V_T^k.$$

- continuity: $a_h^T(u_h,v_h)\le c|u_h|_{1,T}|v_h|_{1,T}, \forall u_h,v_h\in V_T^k$.

Here $|\cdot|_{1,T}$ is H^1 semi-norm on T. The global discrete bilinear form is defined as

$$a_h(u_h,v_h)=\sum_{T\in T_h} a_h^T(u_h,v_h)\mathrm{d}x, \quad \forall u_h,v_h\in V_T^k. \tag{9}$$

We define f_h by

$$<f_h,v_h>=\begin{cases}\sum_{T\in T_h}\int_T(\Pi_{T,k}^0 f)P_0 v_h\mathrm{d}x, \forall v_h\in V_h^k, k=1,\\ \sum_{T\in T_h}\int_T(\Pi_{T,k}^0 f)v_h\mathrm{d}x, \forall v_h\in V_h^k, k\ge 2.\end{cases} \tag{10}$$

The approximate function f_h has the property [8] as

$$(f,v_h)-<f_h,v_h>\le Ch^k\left(\sum_{T\in T_h}|f|_{1,T}^2\right)^{\frac{1}{2}}|v_h|_{1,\Omega}. \tag{11}$$

Lemma 2.2 [16] *Under* **Assumption 2.2**, *there exists constants C related only to k and* ϱ *such that for any* $h, T\in T_h, w\in H^s(T), 2\le s\le k+1$, *there exists an interpolation* w_I *of w in the space* V_T^k *satisfies*

$$\|w-w_I\|_{0,T}+h_T\|w-w_I\|_{1,T}\le Ch_T^s|w|_{s,T}. \tag{12}$$

Based on the constructed virtual element space (6) and the discrete bilinear form (9), we use quadratic virtual element and define the discrete form of the obstacle problem (1) as

$$\begin{cases}\text{Find } u_h\in K_h^2,\text{such that}\\ a_h(u_h,v_h-u_h)\ge(f_h,v_h-u_h), \forall v_h\in K_h^2,\end{cases} \tag{13}$$

where $K_h^2=\{v_h\in V_h^2;\chi_i(v_h)\ge\chi_i(\psi), i=1,\ldots,N^{dof}\}$ and N^{dof} denotes the total number of degrees of freedom. By Fei Wang's *article* [14], the discrete problem (13) is well-posed and has a unique solution $u_h\in K_h^2$.

3. Error estimation

Theorem 3.1. *Let* $u\in W^{s,p}(\Omega)\cap H_0^1(\Omega)$, *where* $1<p<\infty$ *and* $s<2+1/p$. u,u_h *are solutions of the problems* (1) *and* (13) *respectively. Assuming* $\psi\in H^3(\Omega)$, $f\in H^1(\Omega)\cap L^\infty(\Omega)$, *there is*

$$|u-u_h|_{1,\Omega}\le Ch^{3/2-\varepsilon}(|u|_{\frac{5}{2}-\varepsilon,\Omega}+|f|_{\frac{1}{2}-\varepsilon,\Omega}), \tag{14}$$

where $\varepsilon>0$ *and is* related *to* s,p.

Proof. We divide the error into two parts,

$$e=u-u_h=(u-u_I)+(u_I-u_h)=:e_I+e_h. \tag{15}$$

From the stability and consistency of the bilinear form $a(\cdot,\cdot)$ and discrete problem (13), we have

$$
\begin{aligned}
c_s|e_h|_{1,\Omega}^2 &\le a_h(e_h,e_h)=a_h(u_I,e_h)-a_h(u_h,e_h)\\
&\le \sum_{T\in T_h}\left(a_h^T(u_I-u_\pi,e_h)+a_h^T(u_\pi,e_h)\right)-(f_h,e_h)\\
&=\sum_{T\in T_h}\left(a_h^T(u_I-u_\pi,e_h)+a^T(u_\pi-u,e_h)\right)+a(u,e_h)-(f_h,e_h)\\
&=\sum_{T\in T_h}\left(a_h^T(u_I-u_\pi,e_h)+a^T(u_\pi-u,e_h)\right)+(f-f_h,e_h)+\left(a(u,e_h)-(f,e_h)\right)\\
&=:R_1+R_2+R_3.
\end{aligned}
\tag{16}
$$

First, by using the continuity of the bilinear forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$, **Lemma 2.1**, and **Lemma 2.2**, there is

$$
\begin{aligned}
R_1&=\sum_{T\in T_h}\left(a_h^T(u_I-u,e_h)+a_h^T(u-u_\pi,e_h)+a^T(u_\pi-u,e_h)\right)\\
&\le \sum_{T\in T_h} C(|u-u_I|_{1,T}+|u-u_\pi|_{1,T})|e_h|_{1,T}\\
&\le \sum_{T\in T_h} Ch^{\frac{3}{2}-\epsilon}|u|_{\frac{5}{2}-\epsilon,T}|e_h|_{1,T}\\
&\le Ch^{3-2\epsilon}|u|_{\frac{5}{2}-\epsilon,\Omega}^2+\frac{c_s}{4}|e_h|_{1,\Omega}^2.
\end{aligned}
\tag{17}
$$

From the article of F. Dassi et al. [11], we have

$$
\begin{aligned}
R_2=\sum_{T\in T_h}\int_T(f-\bar{f})(e_h-\bar{e}_h)\mathrm{d}x&\le \sum_{T\in T_h}||f-\bar{f}||_{0,T}||e_h-\bar{e}_h||_{0,T}\\
&\le Ch^{\frac{3}{2}-\epsilon}|f|_{\frac{1}{2}-\epsilon,\Omega}|e_h|_{1,\Omega}\\
&\le Ch^{3-2\epsilon}|f|_{\frac{1}{2}-\epsilon,\Omega}^2+\frac{c_s}{4}|e_h|_{1,\Omega}^2,
\end{aligned}
\tag{18}
$$

where $\bar{f}$ means $\frac{1}{|T|}\int_T f\,\mathrm{d}x$, and $\int_T(e_h-\bar{e}_h)\mathrm{d}x=0$.

Finally, in order to estimate R_3, we recall that the region Ω is partitioned into Ω^+ and Ω^0. Then for any element $T\in T_h$, there are three cases, $T\subset\Omega^+$, $T\subset\Omega^0$, and $T\cap\Omega^+\neq\varnothing$ with $T\cap\Omega^0\neq\varnothing$. In the following, T^+ and T^0 denote the two parts of the element T belonging to the regions Ω^+ and Ω^0 respectively, i.e. $T^+=T\cap\Omega^+$, and $T^0=T\cap\Omega^0$. By using the equation (3), one can get

$$
\begin{aligned}
R_3&=\int_\Omega\nabla u\nabla e_h\mathrm{d}x-\int_\Omega f e_h\mathrm{d}x=\int_\Omega(-\Delta u-f)e_h\mathrm{d}x\\
&=\sum_{T\in T_h}\left(\int_{T^+}(-\Delta u-f)e_h\mathrm{d}x+\int_{T^0}(-\Delta u-f)e_h\mathrm{d}x\right)\\
&=\sum_{T\in T_h}\int_{T^0}(-\Delta u-f)e_h\mathrm{d}x\\
&=\sum_{T\in T_h}\int_{T^0}(-\Delta u-f)(\psi_I-u_h)\mathrm{d}x\\
&=\sum_{T\in T_h}\int_T(-\Delta u-f)(\psi_I-u_h)\mathrm{d}x.
\end{aligned}
\tag{19}
$$

Based on the degrees of freedom (D1)-(D3) and the definition of K_h^2, we have

$$
\int_T(\psi_I-u_h)\mathrm{d}x=|T|\left(\chi_{N^{dof}}(\psi_I)-\chi_{N^{dof}}(u_h)\right)=|T|\left(\chi_{N^{dof}}(\psi)-\chi_{N^{dof}}(u_h)\right)\le 0.
$$

From the differential form of the unilateral obstacle problem (2), we have $-\Delta u-f\ge 0$, hence $P_0(-\Delta u-f)=\frac{1}{|T|}\int_T(-\Delta u-f)\mathrm{d}x\ge 0$. Then $-\int_T P_0(-\Delta u-f)(\psi_I-u_h)\mathrm{d}x\ge 0$, and

$$
\begin{aligned}
&\int_T(-\Delta u-f)(\psi_I-u_h)\mathrm{d}x\\
&\leq\int_T((-\Delta u-f)-P_0(-\Delta u-f))\,(\psi_I-u_h)\mathrm{d}x\\
&=\int_T((-\Delta u-f)-P_0(-\Delta u-f))\,\big((\psi_I-u_h)-P_0(\psi_I-u_h)\big)\mathrm{d}x\\
&=\int_{T^0}((-\Delta u-f)-P_0(-\Delta u-f))\,((u_I-u_h)-P_0(u_I-u_h))\mathrm{d}x\\
&=\int_{T^0}((-\Delta u-f)-P_0(-\Delta u-f))\,(e_h-P_0e_h)\mathrm{d}x\\
&\leq\|(-\Delta u-f)-P_0(-\Delta u-f)\|_{0,T^0}\|e_h-P_0e_h\|_{0,T^0}\\
&\leq\|(-\Delta u-f)-P_0(-\Delta u-f)\|_{0,T}\|e_h-P_0e_h\|_{0,T}\\
&\leq Ch^{\frac{3}{2}-\epsilon}|-\Delta u-f|_{\frac{1}{2}-\epsilon,T}|e_h|_{1,T}.
\end{aligned}\tag{20}
$$

Substituting (20) into (19), we get

$$R_3\leq\sum_{T\in\mathrm{T}_h}c\,h^{\frac{3}{2}-\epsilon}|-\Delta u-f|_{\frac{1}{2}-\epsilon,T}|e_h|_{1,T}\leq Ch^{3-2\epsilon}|-\Delta u-f|^2_{\frac{1}{2}-\epsilon,\Omega}+\frac{c_s}{4}|e_h|^2_{1,\Omega}. \tag{21}$$

Substituting (17), (18) and (21) into (16), it holds that

$$|e_h|_{1,\Omega}\leq Ch^{\frac{3}{2}-\epsilon}(|u|_{\frac{5}{2}-\epsilon,\Omega}+|f|_{\frac{1}{2}-\epsilon,\Omega}). \tag{22}$$

Finally, we accomplish the proof by triangle inequality, (22) and Lemma 2.2.

4. Numerical experiments

The obstacle problem (1) is considered in the domain $\Omega=(-2,2)^2$ with the right hand side function $f(x,y)=0$ and obstacle function

$$\psi(x,y)=\begin{cases}\sqrt{1-r^2}, & r\leq 1,\\ -1, & r>1,\end{cases}$$

where $r=\sqrt{x^2+y^2}$. The boundary condition is determined from the exact solution

$$u(x,y)=\begin{cases}\sqrt{1-r^2}, & r\leq r^*,\\ -(r^*)^2\ln(r/2)/\sqrt{1-(r^*)^2}, & r\geq r^*,\end{cases}$$

where r^* is the root of the equation $(r^*)^2(1-\ln(r^*/2))=1$, and $r^*\approx 0.6979651482$ by calculation. The obstacle function $\psi(x,y)$ and exact solution $u(x,y)$ are shown in Figure 1. We perform numerical experiments on convex and non-convex polygonal meshes with $k=2$ (see Figure 2).

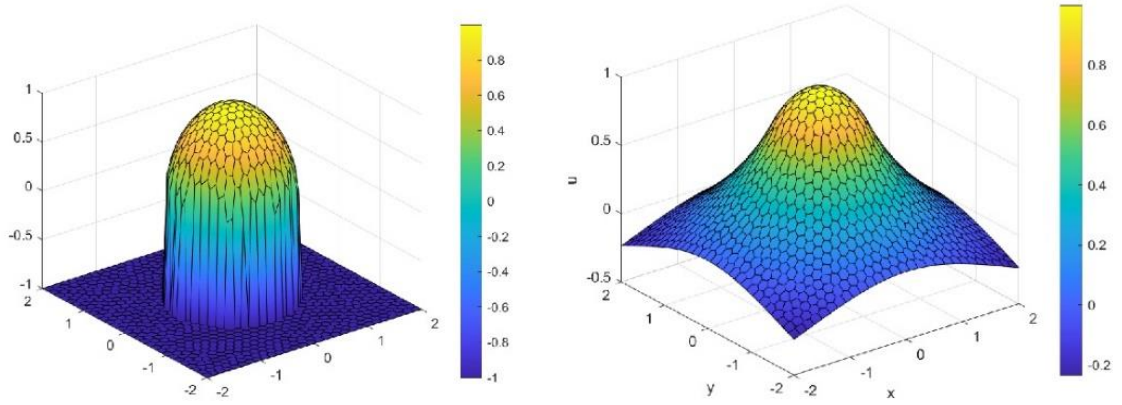

Figure 1. $\psi(x,y)$ and $u(x,y)$.

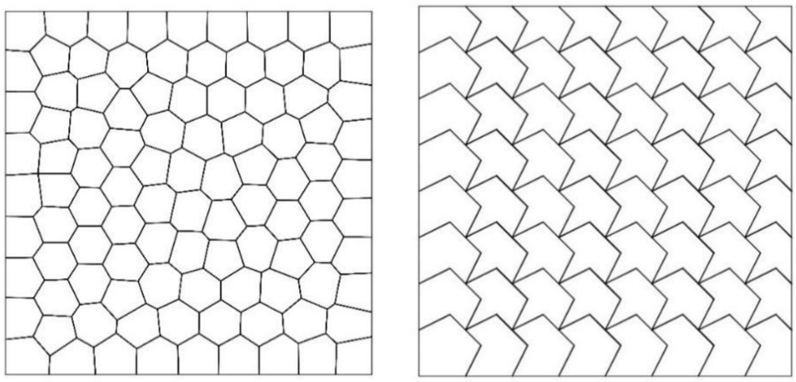

Figure 2. Polygonal mesh and non-convex mesh.

The numerical results are shown in Table 1 and Table 2, where h means the size of the meshes and projection operator Π_2 is defined by the relation $\Pi_2|_T=\Pi_{T,2}$. As can be seen from the tables, the convergence order of the numerical solutions in both meshes is about 1.5, which verifies the theoretical result of **Theorem 3.1.**

Table 1. Errors on convex polygonal meshes.

h	$\|u-\Pi_2 u_h\|_1$	
0.342057	1.690894e-01	-
0.179451	5.829676e-02	1.650792
0.087447	2.127046e-02	1.402520
0.062864	1.234701e-02	1.647907

Table 2. Errors on non-convex polygonal meshes.

h	$\|u-\Pi_2 u_h\|_1$	
0.745356	2.343676e-01	-
0.372678	8.977114e-02	1.309421
0.186339	3.249997e-02	1.330847
0.093169	1.099309e-02	1.563841

5. Conclusion

The obstacle problem is a typical issue in contact mechanics, and its numerical solution is of great significance. Virtual element methods have advantages in solving such problems, but the theoretical analysis is somewhat challenging. This paper presents an convenient theoretical analysis framework for the analysis of approximate error, which has certain theoretical reference value for similar issues. The numerical experimental on the concave grid also confirm that the virtual element has more flexibility in meshing compared to the finite element method.

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