

**Research on Anti-Hermitian Problem of the Reduced Biquaternion Matrix Equation** $AX + XB = C$

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**Abstract:** We study the anti-Hermitian problem of reduced biquaternion matrix equation $AX + XB = C$. We mainly search for the sufficient and prerequisite condition for the solution.

### 1. Introduction

Throughout this paper, we discuss the anti-Hermitian reduced biquaternion problem of equation

$$AX + XB = C.$$  \hfill (1)

An obvious difference between quaternions and reduced biquaternion is that reduced biquaternions are commutative, whereas quaternions form a skew field. Thus, many properties of reduced biquaternion are understandable and reduced biquaternion matrices can represent color images [1, 2].

There are some interpretation of symbols, $R^m$, $R^{m\times n}$, $SR^{n\times n}$ and $ASR^{m\times n}$ denote all real number, $m\times n$ matrices, $n\times n$ symmetric matrices, and $n\times n$ anti-symmetric matrices. $C^{m\times n}$ denotes all $m\times n$ complex matrices. $Q_{BB}$, $Q_{B R}$, $Q_{R B}$ denote all reduced biquaternions, $n$-dimensional reduced bi quaternion column vectors, $m\times n$ reduced biquaternion matrices, respectively. For $A \in C^{m\times n}$, $\text{Im}(A)$ is imaginary part of matrix $A$, $\text{Re}(A)$ is real part of matrix $A$. For $A \in Q_{BB}^{m\times n}$, $\overline{A}$ is conjugate, $A^T$ is transpose, and $A^H$ is conjugate and transpose. $I_n$ denotes the identity matrix, $0_{m\times n}$ denotes the zeros matrix. For $A \in R^{m\times n}$, the Moore-Penrose inverse $A^+$, satisfies

$$AXA = A, XAX = X, (AX)^T = AX, (XA)^T =XA.$$  

A matrix $A \in Q_{BB}^{m\times n}$ is anti-Hermitian if $A^H = -A$. Denote the anti-Hermitian reduced biquaternion matrix by $AHQ_{BB}^{m\times n}$. The problem of this paper is as follows.
Problem A. Suppose $A \in Q_{RB}^{n \times n}$, $B \in Q_{RB}^{m \times n}$, and $C \in Q_{RB}^{m \times n}$. Find $H_E = \{ X \mid X \in A H Q_{RB}^{n \times n}, AX + XB = C \}$. \hfill (2)

We will search for $H_E$ to be nonempty. We will obtain a general expression and a unique solution for $H_E$.

Quaternions was introduced in 1843 [3]. The properties and results can be found in [4,5]. The reduced biquaternions was introduced by Schütte and Wenzel [6]

$$Q_{RB} = \{ q \mid q = q_0 + q_i i + q_j j + q_k k, q_0, q_i, q_j, q_k \in \mathbb{R} \},$$

where

$$i^2 = k^2 = -1, j^2 = 1, ij = ji = k, jk = kj = i, ik = ki = -j.$$

A reduced biquaternion is $q = q_0 + q_i i + q_j j + q_k k$ or $q = c_1 + c_2 j$, where $c_1 = q_0 + q_i$ and $c_2 = q_2 + q_3 i$ are complex numbers.

The complex expression for $q = c_1 + c_2 j$ is

$$f(q) = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} \in \mathbb{C}^{2 \times 2}.$$ \hfill (3)

For $a, b \in Q_{RB}$, note that $f(ab) = f(a)f(b)$. Similarly, $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{m \times n}$. The complex representation matrix of $A = A_1 + A_2 j \in Q_{RB}^{n \times n}$ is

$$F(A) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \in \mathbb{C}^{2m \times 2n},$$ \hfill (4)

and $F(A)$ is unique. For $E \in Q_{RB}^{n \times m}$, $H \in Q_{RB}^{m \times n}$, we have

$$F(EH) = F(E)F(H).$$ \hfill (5)

For $A = (a_{ij}) \in Q_{RB}^{m \times n}$, $B = (b_{ij}) \in Q_{RB}^{n \times p}$, let $A \otimes B = (a_{ij}b_{ij}) \in Q_{RB}^{m \times p}$. For $A = (a_{ij}) \in Q_{RB}^{n \times m}$, $a_j = (a_{1j}, a_{2j}, \ldots, a_{nj})$, where $j = 1, 2, \ldots, n$. Define

$$\text{vec}(A) = (a_1, a_2, \ldots, a_n)^T.$$ \hfill (6)

2. The Structure of vec$(ABC)$ and vec$(AXB)$

For any $q = c_1 + c_2 j \in Q_{RB}$, we indicate the symbol $\cong$, it is,

$$c_1 + c_2 j = q \cong \Phi_q = (c_1, c_2).$$ \hfill (7)

For $E = E_1 + E_2 j \in Q_{RB}^{n \times n}$, we have

$$E_1 + E_2 j = E \cong \Phi E = [E_1, E_2].$$ \hfill (8)
Addition of $E = E_i + E_j$ and $D = D_i + D_j$ is 
\[(E_i + D_i) + (E_j + D_j) = E + D = \Phi_{E,D} = [E_i + D_i, E_j + D_j].\]

Similarly, multiplication of $E = E_i + E_j$ and $D = D_i + D_j$ is 
\[ED = (E_i + E_j)(D_i + D_j) = (E_iD_i + E_iD_j + E_jD_i + E_jD_j)j.\]

So $ED \cong \Phi_{ED}$.

Theorem 2.1. Suppose $c \in R$, $q \in Q_{RB}, E, D \in \mathbb{Q}_{RB}^{m \times n}$, $H \in Q_{RB}^{n \times t}$, $A = (a_{ij})_{m \times n}$, where $a_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)}$. Then

(i) $E = D$ if and only if $\Phi_E = \Phi_D$;
(ii) $\Phi_{E+D} = \Phi_E + \Phi_D$, $\Phi_{cE} = c\Phi_E$;
(iii) $\Phi_{EH} = \Phi_E F(H)$.

Theorem 2.2. Suppose $E = E_i + E_j \in \mathbb{Q}_{RB}^{m \times n}$, $D = D_i + D_j \in \mathbb{Q}_{RB}^{n \times s}$, and $H = H_1 + H_2 \in \mathbb{Q}_{RB}^{n \times t}$, where $E_i, E_j \in C_{mn}^{m \times n}, D_i, D_j \in C_{nm}^{n \times m}$, and $H_1, H_2 \in C_{nt}^{n \times t}$. Then

\[
vec(\Phi_{EDH}) = F[(H_1^T \otimes E_1 + H_2^T \otimes E_2) + (H_2^T \otimes E_1 + H_1^T \otimes E_2)]j \begin{bmatrix} vec(D_1) \\ vec(D_2) \end{bmatrix}.
\] (9)

Proof. The proof is resemble to that in Lemma 4 in the reference [9], so we omit it. The conclusion of $vec(\Phi_{ABC})$ is an approach for solving Problem I.

Definition 2.3. For $D = (d_{ij}) \in \mathbb{Q}_{RB}^{m \times n}$, let 
\[d_1 = (d_{11}, \sqrt{2}d_{21}, \ldots, \sqrt{2}d_{1n}), d_2 = (d_{22}, \sqrt{2}d_{32}, \ldots, \sqrt{2}d_{2n}), \ldots, d_{n-1} = (d_{(n-1)x(n-1)}, \sqrt{2}d_{n(n-1)}), d_n = d_{nn},\]
and 
\[vec_3(D) = (d_1, d_2, \ldots, d_{n-1}, d_n)^T \in Q_{RB}^{n(n+1)/2}.\] (10)

Definition 2.4. For $H = (h_{ij}) \in \mathbb{Q}_{RB}^{n \times n}$, let 
\[h_1 = (h_{21}, h_{31}, \ldots, h_{1n}), h_2 = (h_{32}, h_{42}, \ldots, h_{2n}), \ldots, h_{n-2} = (h_{(n-1)x(n-1)}, h_{n(n-2)}), h_{n-1} = h_{n(n-1)},\]
and 
\[vec_A(H) = \sqrt{2}(h_1, h_2, \ldots, h_{n-2}, h_{n-1})^T \in Q_{RB}^{n(n-1)/2}.\] (11)

Lemma 2.5 [7]. For $Y \in \mathbb{R}_{n \times n}$, one gets

(1) $Y \in S\mathbb{R}_{n \times n} \Leftrightarrow vec(Y) = K_S vec_3(Y)$,

where $vec_3(Y)$ is explained by (10), and $K_S \in \mathbb{R}_{n \times (n+1)/2}$ is
\[
K_S = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2}e_1 & e_2 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & e_1 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_n & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & e_1 & 0 & 0 & 0 & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\
0 & 0 & 0 & e_1 & 0 & 0 & \cdots & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n
\end{bmatrix}.
\]

(13)

(2) \( Y \in ASR^{n,n} \Leftrightarrow \text{vec}(Y) = K_A \text{vec}_d(Y) \), \( \text{vec}_d(Y) \) is expressed as (11), and the matrix \( K_A \in R^{n^2,n(n-1)/2} \) is

\[
K_A = \frac{1}{\sqrt{2}} \begin{bmatrix}
e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 \\
-e_2 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n \\
0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\
0 & 0 & -e_1 & 0 & 0 & 0 & -e_2 & 0 & \cdots & e_n \\
0 & 0 & 0 & -e_1 & 0 & 0 & \cdots & -e_2 & \cdots & -e_{n-1}
\end{bmatrix},
\]

(14)

where \( e_i \) is the \( i \)th column of \( I_n \).

For any \( X = X_1 + X_2 j \in Q_{RB}^{n,n} \), we have

\[
X \in AHQ_{RB}^{n,n} \Leftrightarrow \begin{cases}
\text{Re}(X_1)^T = -\text{Re}(X_1), & \text{Im}(X_1)^T = \text{Im}(X_1) \\
\text{Re}(X_2)^T = \text{Re}(X_2), & \text{Im}(X_2)^T = \text{Im}(X_2)
\end{cases}
\]

(15)

Obviously, \( \text{Re}(X_1) \) is anti-symmetric, \( \text{Re}(X_2), \text{Im}(X_1) \) and \( \text{Im}(X_2) \) are symmetric.

3. The Solution of Problem A

Through the above conclusion, we now resolve Problem A. We also need the following lemma.

Lemma 3.1 [8]. The equation \( Ax = b \), with \( A \in R^{m \times n} \) and \( b \in R^n \), has a solution \( x \in R^n \Leftrightarrow AA^Tb = b \).

(16)

Under the circumstance,

\[
x = A^Tb + (I_n - A^TA)z,
\]

(17)

where \( z \in R^n \) is a vector. It has a unique solution \( x = A^Tb \) when \( \text{rank}(A) = n \).

Theorem 3.2. Given \( A = A_1 + A_2 j \in Q_{RB}^{n,n}, B = B_1 + B_2 j \in Q_{RB}^{n,n}, \) and \( C \in Q_{RB}^{n,n} \), let
\[
T = \begin{bmatrix}
I \otimes A_1 + B_1^T \otimes I & I \otimes A_2 + B_2^T \otimes I \\
I \otimes A_2 + B_2^T \otimes I & I \otimes A_1 + B_1^T \otimes I
\end{bmatrix}
\begin{bmatrix}
K_A & iK_S & 0 & 0 \\
0 & 0 & K_S & iK_s
\end{bmatrix},
\]

\[
T_1 = \text{Re}(T), \ T_2 = \text{Im}(T), \ l = \begin{bmatrix}
\text{vec}(\text{Re}(C_1)) \\
\text{vec}(\text{Re}(C_2)) \\
\text{vec}(\text{Im}(C_1)) \\
\text{vec}(\text{Im}(C_2))
\end{bmatrix}.
\]

Problem A has a solution

\[
X = X_1 + X_2, j \in H_E \iff \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} l = l . \tag{18}
\]

When the condition satisfies, then

\[
\begin{bmatrix}
\text{vec}_A(\text{Re}(X_1)) \\
\text{vec}_S(\text{Im}(X_1)) \\
\text{vec}_S(\text{Re}(X_2)) \\
\text{vec}_S(\text{Im}(X_2))
\end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ z + \begin{bmatrix}
I_{2n^2+n} & -\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^+ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}
\end{bmatrix} z. \tag{19}
\]

Proof. According to Theorem 2.2 and Lemma 2.5, we get

\[
AX + XB = C \iff \Phi_{AX} + \Phi_{XB} = \Phi_{C}
\]

\[
\iff \begin{bmatrix}
I \otimes A_1 + B_1^T \otimes I & I \otimes A_2 + B_2^T \otimes I \\
I \otimes A_2 + B_2^T \otimes I & I \otimes A_1 + B_1^T \otimes I
\end{bmatrix}
\begin{bmatrix}
\text{vec}(X_1) \\
\text{vec}(X_2)
\end{bmatrix} = \begin{bmatrix}
\text{vec}(C_1) \\
\text{vec}(C_2)
\end{bmatrix}
\]

\[
\iff \begin{bmatrix}
\text{vec}_A(\text{Re}(X_1)) \\
\text{vec}_S(\text{Im}(X_1)) \\
\text{vec}_S(\text{Re}(X_2)) \\
\text{vec}_S(\text{Im}(X_2))
\end{bmatrix} = l.
\]

According to Lemma 3.1, the sufficient and prerequisite condition of the equation (1) existing a solution is (18). If (18) satisfies, then we get (19).

Conclusions

In this article, we discussed \( AX + XB = C \) by the above methods. The least squares problem is a hard work for us to research in future.

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References