The pricing of a venture capital contract with double barriers under Kou's model

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Abstract: Venture capital contract plays a key role in venture capital investment. In order to ensure the fairness of venture capitalists and venture entrepreneurs, a venture capital contract with double barrier option and bond features was designed in this paper. By setting the two barriers, the incentives and controls were established on both parties. Under the assumption that the enterprise value follows Kou's double exponential jump-diffusion process, the pricing formula of venture capital contract and the relationship among the parameters in the contract were obtained by the risk-neutral method.

1. Introduction

Venture capital investment is a hot implement for the funding of start-up companies with high technology and great growth potential, it is an irreversible, high-risk and high-return investment because of the high uncertainties. One big challenge of venture investment is to decide when to release the fund and when to exit[1]. The venture capital investment contract is designed to mitigate the uncertainty and solve the problem, and a good contract can establish the incentives and constraints to both the venture capitalists and venture entrepreneurs[2]. Thus, the evaluation of the venture capital contract plays an important role in venture capital.

With the development of venture capital, a lot of researches has been conducted to price the venture capital contract. One of the traditional methods is the net present value (NPV) method which aims at calculating the present value of the future cash flows of the company, but this method has some defects: it neglects the dynamical risk and the manager's ability to face the huge changes and regards all the uncertainty as the cash outflows[3]. To describe the dynamical change of the company value, there are researches about the application of option pricing on venture capital decision-making, for example, the call option pricing formula under the classic Black-Scholes model can be used to price venture capital contract[4]. However, the model is based on a lot of assumptions, such as the market is complete and no dividends were paid during the process, so the model does not work very well when applied to the real market.

In this paper, we consider a venture investment contract with double barrier features under the Kou's double exponential jump-diffusion model, the double barriers can help to make incentives and control on capitalists and entrepreneurs. The double exponential jump-diffusion model takes account of the jump of the company value so it can coincide with the change of the company value better in the real market. Then, we also find the distribution of the first passage time of the barriers. Finally, we can price this investment contract according to the distribution we found.

2. A VC contract with double barrier features under the Kou's model

It is important to choose suitable financial instruments to design a venture capital contract. In general, there are 5 widely-used financial instruments, which are common stock, preferred stock, common bond, convertible bond and subscribed stock bond. We will choose the convertible bond since it combines the advantages of the common stock and common bond. The convertible bond can be converted to the common bond and enables the venture capitalist to obtain great returns, but we need to decide when to convert the bond.

It is not difficult to see that the venture capital is full of uncertainty and the venture capital contract value will vary as the company value varies, this is similar with the feature of the path-dependent option, whose value will vary as the value of the underlying asset varies during the option's life. So we can design a venture capital contract like a double barrier option, which is one of the path-dependent options. The value of the double barrier option will depend on whether the underlying asset price attained or passages the upper barrier or the lower barrier, and the two barriers are some specified prices. We can use the double barrier feature to help capitalists to decide when to exercise the conversion right and the liquidation right.

2.1 The content of the venture capital contract with double barrier features

We design a venture capital contract as follows: At time $t = 0$, the capitalist invests the funding V_0 into the start-up company. And the venture capitalists and venture entrepreneurs agree that during the life of the contract $[0, T]$:

(a) If the company value $S(t)$ attains or exceeds the upper barrier H for the first time, the venture capitalist will exercise the conversion right at that time;

(b) If the company value hits or goes below the lower barrier h, the venture capitalist will exercise the liquidation right at that time;

(c) If the company value does not go beyond the two barriers, the venture capitalist will receive the principal and the interest at the maturity of the contract.

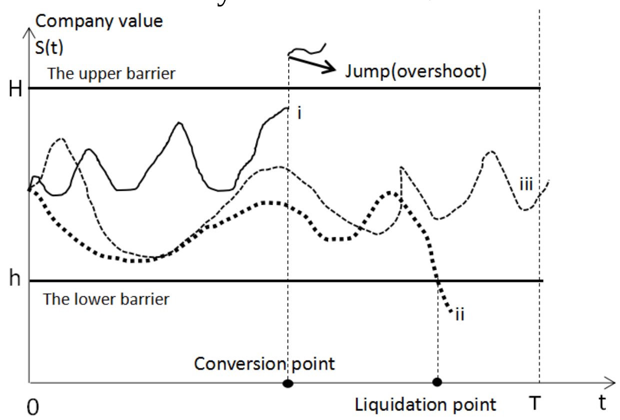

Fig 1: The working mechanism of the venture contract

From the figure above, we can see that, similar to the double barrier option, the venture capital contract we design also has an upper barrier and a lower barrier. The upper barrier can be viewed as a rational boundary with the highest return, and the lower barrier can be viewed as a safe boundary with the lowest return. Besides, the real line shows the situation (a) when the company value exceeds the upper barrier after the overshoot of the company value, then the convertible bond will be converted to common bond; the dotted line shows the situation (b) when the company value exceeds the lower barrier, then the liquidation right will be used for capital to exit; the dashed shows the situation (c) when the company value keeps fluctuating between the two barriers, the basic feature of the common bond will be preserved and the capitalists will receive the face value and the interests of the bond as their return[5].

2.2 The features of the venture capital contract

From the perspective of venture capitalists, this contract can help them obtain upside participation and downside protection. According to the contract: if the company value increases above the upper barrier, the capitalist's conversion right is a kind of equity anti-dilution protection; if the company value decreases below the lower barrier, the capitalist's liquidation right is a kind of downside protection for them to release fund timely; if the company value goes up and down between the two barriers, the capitalist will retain the liquidation and conversion right, so they will not miss the companies with uncertain prospects.

From the perspective of venture entrepreneurs, this contract can provide them with enough development opportunities and retained options because capitalists are not allowed to release funds unless the company value hits or passages the lower barrier. Besides, this contract also helps venture entrepreneurs to increase the real option value: if the venture capitalists disobey the rules agreed with venture entrepreneurs before or fail to meet venture entrepreneurs' expectations, the venture entrepreneurs can let the company value keep between the two barriers, so entrepreneurs still take control of their company and can look for new suitable capitalists; if venture capitalists do well and entrepreneurs believe the capitalists can help the company grow up well, then the venture entrepreneurs will work hard to increase the company value to exceed the upper barrier.

2.3 Kou's model

To price the venture capital contract, we also need to find a suitable model to simulate the change of company value. In this paper, we assume that the company value follows Kou's double exponential jump-diffusion process (2002). Kou's model takes the jump part into consideration, which stands for the discontinuous effect of external information on company value. In the real market, the jump exits because of some sudden news and policies, so Kou's model can coincide with the real data better in the financial market.

The main idea of the model is very simple, it assumes that the stock price follows a Brownian motion by the addition of a jump process, the occurrence of jump follows a Poisson process, and the log jump size follows a double exponential distribution. In our case, under the physical probability measure, the company value $S(t)$ satisfies the following stochastic differential equation[6]:

$$\frac{dS(t)}{S(t-)} = (\mu + \delta)dt + \sigma dW(t) + d(\sum_{i=1}^{N(t)}(V_i - 1))$$

where $(\mu + \delta)dt$ is the drift term, μ denotes the natural growth rate of venture company value, δ denotes the contribution rate of venture capitalists, W_t is a standard Brownian motion, $\{N_t, t \geq 0\}$ is a Poisson process with intensity λ, the jump size $\{V_k\}, k = 1,2...$ is a sequence of i.i.d non-negative random variables, and the log jump size Y=log(V) has double exponential distribution. The density function of double exponential distribution is given by: $f_Y(y) = p_u\eta_u e^{-\eta_u y} I_{[y\geq 0]} + p_d\eta_d e^{-\eta_d y} I_{[y<0]}$, where $\eta_u > 1$, $\eta_d > 0$ are some constants, p_u stands for the probability of upward jump, p_d represents the probability of downward jump, and $p_u + p_d = 1$.

3. Pricing the venture contract with double barrier features

The pricing of venture capital contract is a key step in venture capital. By evaluating the contact before the investment, venture capitals and venture entrepreneurs can make adjustments of the parameters in the contract to achieve the fairness between them. According to the contract, there are three situations during the life of the venture investment:(a)The company value $S(t)$ attains or exceeds the upper barrier H; (b)The company value hits or goes below the lower barrier h; (c)The company value does not go beyond of the two barriers. In this contract, we define the situations are mutually exclusive, so we need to find the probability and the return under each situation.

Suppose τ_H is the first passage time of the upper barrier H, and τ_h is the first passage time of the lower barrier h. In other words, $\tau_H = \{t \geq 0, S(t) \geq H\}, \tau_h = \{t \geq 0, S(t) \leq h\}$, then we can describe the above mutually exclusive events as follow:

$$\text{(a)}A = \{\tau_H < T\}; \text{(b)}B = \{\tau_h < T\}; \text{(c)}C = \{\tau_H \geq T\} \cup \{\tau_h \geq T\}$$

Then, the expected return of this venture capital contract can be divided into 3 parts:

$$E[R(S(T))] = E[I_A R_A(S_A(T))] + E[I_B R_B(S_B(T))] + E[I_C R_C(S_C(T))] \qquad (1)$$

where I denotes the indicative function of each event, R denotes the return function of different company value at different time.

Note that the equation (1) only can be calculated under the risk-neutral measure, so we need to find the process of company value under the risk-neutral measure. Kou(2002) proposed that under the risk-neutral probability measure P^*, the company value still follows a Brownian motion plus a jump process:

$$\frac{dS(t)}{S(t-)} = (r + \delta - \lambda^*\xi^*)dt + \sigma dW^*(t) + d(\sum_{i=1}^{N^*(t)} (V_i^* - 1))$$

where r is the risk-free rate of interest, ξ^* is the mean percentage jump size, which is defined as: $\xi^* = E[V_*] - 1 = E[e^{Y^*}] - 1$, and the parameters and processes with "*" denotes the counterpart under the new probability measure P^*.

For notation simplicity, we drop all the notations "*" in the following discussion, and all the parameters and processes in the following discussion are under P^*. The Laplace exponent of this process is calculated as follows:

$$\begin{aligned} G(x) &= \frac{1}{2}\sigma^2 x^2 + (r - \frac{\sigma^2}{2} - \lambda\xi)x + \lambda(E[e^{xY}] - 1) \\ &= \frac{1}{2}\sigma^2 x^2 + (r - \frac{\sigma^2}{2} - \lambda\xi)x + \lambda(\frac{p_u\eta_u}{\eta_u - x} + \frac{p_d\eta_d}{\eta_d + x} - 1) \end{aligned}$$

Consider the equation $G(x) = \alpha, \forall\alpha > 0$, it has two positive real roots $\gamma_{1,\alpha}, \gamma_{2,\alpha}$ and two negative real roots $-\gamma_{3,\alpha}, -\gamma_{4,\alpha}$, which satisfy: $0 < \gamma_{1,\alpha} < \eta_u < \gamma_{2,\alpha} < \infty, 0 < \gamma_{3,\alpha} < \eta_d < \gamma_{4,\alpha} < \infty$. Then, we can calculate three expectations in equation (1) respectively.

3.1 Company value first passages the upper barrier

To calculate the first expectation on the right side of equation (1), which stands for the return under situation (a) when the company value first passages the upper barrier, we need to find the distribution of the first passage time of the upper barrier. Under Kou's model, the distribution of the first passage times can be obtained by Laplace transforms. Kou and Wang (2003) gave the explicit solution of the Laplace transform of the first passage time τ_H of the upper barrier[7]:

$$m(\gamma_{1,\alpha}, \gamma_{2,\alpha}, H) = E[e^{-\alpha\tau_H}] = \frac{\eta_u - \gamma_{1,\alpha}}{\eta_u}\frac{\gamma_{2,\alpha}}{\gamma_{2,\alpha} - \gamma_{1,\alpha}}e^{-H\gamma_{1,\alpha}} + \frac{\gamma_{2,\alpha} - \eta_u}{\eta_u}\frac{\gamma_{1,\alpha}}{\gamma_{2,\alpha} - \gamma_{1,\alpha}}e^{-H\gamma_{2,\alpha}}$$

where the parameters are defined the same as before. And note that

$$\int_0^\infty e^{-\alpha t}P(\tau_H \le t)dt = \frac{1}{\alpha}\int_0^\infty e^{-\alpha t}dP(\tau_H \le t) = \frac{1}{\alpha}E[e^{-\alpha\tau_H}]$$

To find out the distribution of the first passage time, we need to calculate the Laplace inversion. We will use the Gaver-Stehfest algorithm to find the solution of Laplace inversion in this paper. The main idea of the algorithm is given as follows[8]:

$$u(t) = \lim_{n\to\infty} u_n(t), \text{ and } u_n(t) = \frac{(2n)!\ln 2}{n!(n-1)!t}\sum_{k=0}^{n}(-1)^k C_n^k\,\hat{u}(\frac{(n+k)\ln 2}{t}) \quad (2)$$

where $\hat{u}$ is the Laplace transform of the original function $u(t)$. In our case, $\hat{u}(\alpha)$ is given as:

$$\hat{u}(\alpha) = \frac{1}{\alpha}E[e^{-\alpha\tau_H}] = \frac{\gamma_{2,\alpha}(\eta_u - \gamma_{1,\alpha})}{\alpha\eta_u(\gamma_{2,\alpha} - \gamma_{1,\alpha})}e^{-H\gamma_{1,\alpha}} + \frac{\gamma_{1,\alpha}(\gamma_{2,\alpha} - \eta_u)}{\alpha\eta_u(\gamma_{2,\alpha} - \gamma_{1,\alpha})}e^{-H\gamma_{2,\alpha}}$$

Substituting the above equation into (2) gives the cumulative probability function of τ_H:

$$P(\tau_H \le t) = \lim_{n\to\infty}\frac{(2n)!\ln 2}{n!(n-1)!t}\sum_{k=0}^{n}(-1)^k C_n^k\frac{t}{(n+k)\ln 2}E[e^{-\frac{(n+k)\ln 2}{t}\tau_H}] \quad (3)$$

Then, we can calculate the expected return under situation (a). Suppose ω is the proportion of the capitalist's shares after exercising the conversion right, the expectation can be expressed as:

$$E[I_A R_A(S_A(T))] = E[I_A \omega H e^{(r+\delta)(T-\tau_H)}]$$

where δ is the contribution rate of venture capitalists defined as the contribution rate of venture capitalists of the company. Since the indicative function and the return function are dependent, we cannot directly take the expectation of the indicative function I_A as $P(A)$, instead, we need to calculate the density probability function of the first passage time τ_H, in other words, we should take the derivative of the equation (3) with respect to t. The density probability function is $f_A(t) = \frac{\partial}{\partial t} P(\tau_H \le t)$, then

$$E[I_A R_A(S_A(T))] = \int_0^T f_A(t)\, \omega H e^{(r+\delta)(T-t)} dt = \omega H e^{(r+\delta)T} \int_0^T e^{-(r+\delta)t} \frac{\partial P(\tau_H \le t)}{\partial t} dt$$

3.2 Company value first passages the lower barrier

Similarly, we can also obtain the Laplace transform of the first passage time τ_h of the lower barrier and its cumulative probability function:

$$n(\gamma_{3,\alpha}, \gamma_{4,\alpha}, h) = E[e^{-\alpha\tau_h}] = \frac{\eta_d - \gamma_{3,\alpha}}{\eta_d} \frac{\gamma_{4,\alpha}}{\gamma_{4,\alpha} - \gamma_{3,\alpha}} e^{h\gamma_{3,\alpha}} + \frac{\gamma_{4,\alpha} - \eta_d}{\eta_d} \frac{\gamma_{4,\alpha}}{\gamma_{4,\alpha} - \gamma_{3,\alpha}} e^{h\gamma_{4,\alpha}}$$

$$P(\tau_h \le t) = \lim_{n\to\infty} \frac{(2n)!\ln 2}{n!(n-1)!t} \sum_{k=0}^{n} (-1)^k C_n^k \frac{t}{(n+k)\ln 2} E[e^{-\alpha\tau_h}], \text{ where } \alpha = \frac{(n+k)\ln 2}{t}. \quad (4)$$

Under situation (b) when the company value hits or passages the lower barrier h, the capitalist will exercise the liquidation right. Suppose the density probability function is $f_B(t) = \frac{\partial}{\partial t} P(\tau_h \le t)$, then the corresponding return is given by:

$$E[I_B R_B(S_B(T))] = E[I_B h e^{r(T-\tau_h)}] = \int_0^T f_B(t) h e^{r(T-t)} dt = h e^{rT} \int_0^T e^{-rt} \frac{\partial P(\tau_h \le t)}{\partial t} dt$$

where $P(\tau_h \le t)$ is calculated as (4).

3.3 Company value is between the two barriers

When the company value keeps going up and down between the two barriers, the capitalist will receive the principal he invested and the expected interest, which is $V_0 e^{\rho T}$, where ρ denotes the required return rate when the company develops well. Since the return is a constant, not a function with respect to t, and the expectation of the indicative function is the probability of the event, which is $1 - P(I_A) - P(I_B)$, then the expected return can be expressed as follows:

$$E[I_C R_C(S_C(T))] = E[I_C V_0 e^{\rho T}] = V_0 e^{\rho T} P(I_C) = V_0 e^{\rho T}(1 - P(\tau_H \le T) - P(\tau_h \le T))$$

3.4 The contract value

To sum up, the expected return of the venture contract with double barriers under the Kou's model is given by:

$$\begin{aligned} E[R(S(T))] &= E[I_A R_A(S_A(T))] + E[I_B R_B(S_B(T))] + E[I_C R_C(S_C(T))] \\ &= \omega H e^{(r+\delta)T} \int_0^T e^{-(r+\delta)t} \frac{\partial P(\tau_H \le t)}{\partial t} dt + h e^{rT} \int_0^T e^{-rt} \frac{\partial P(\tau_h \le t)}{\partial t} dt \quad (5) \\ &\quad + V_0 e^{\rho T}(1 - P(\tau_H \le T) - P(\tau_h \le T)) \end{aligned}$$

where

$$P(\tau_H \le t) = \lim_{n\to\infty} \frac{(2n)!\ln 2}{n!(n-1)!t} \sum_{k=0}^{n} [(-1)^k C_n^k \frac{t}{(n+k)\ln 2} (\frac{\eta_u - \gamma_{1,\alpha}}{\eta_u} \frac{\gamma_{2,\alpha}}{\gamma_{2,\alpha} - \gamma_{1,\alpha}} e^{-H\gamma_{1,\alpha}} + \frac{\gamma_{2,\alpha} - \eta_u}{\eta_u} \frac{\gamma_{1,\alpha}}{\gamma_{2,\alpha} - \gamma_{1,\alpha}} e^{-H\gamma_{2,\alpha}})],$$

$$P(\tau_h \le t) = \lim_{n\to\infty} \frac{(2n)!\ln 2}{n!(n-1)!t} \sum_{k=0}^{n} [(-1)^k C_n^k \frac{t}{(n+k)\ln 2} (\frac{\eta_d - \gamma_{3,\alpha}}{\eta_d} \frac{\gamma_{4,\alpha}}{\gamma_{4,\alpha} - \gamma_{3,\alpha}} e^{h\gamma_{3,\alpha}} + \frac{\gamma_{4,\alpha} - \eta_d}{\eta_d} \frac{\gamma_{4,\alpha}}{\gamma_{4,\alpha} - \gamma_{3,\alpha}} e^{h\gamma_{4,\alpha}})].$$

and $\alpha = \frac{(n+k)\ln 2}{t}$.

We obtain the relationship between the contract value and the contract terms by the above equation, so we can use this relationship to make incentives and controls on venture capitalists and venture entrepreneurs by adjusting the terms in the contract. To be specific, it is not difficult to see that the expected value will increase as δ and ρ are increasing, so if the venture capitalists disobey the rules or fail to meet the expectation of venture entrepreneurs, the venture entrepreneurs can choose lower required return rate ρ to ensure that venture capitalists will not receive excess profits and the value of venture contract will be lower with lower ρ; if the venture entrepreneurs work well, venture capitalists can choose higher contribution rate δ to the company to make the company value passages the upper barrier, which enlarges the return of venture capitalists. Besides, we can use this relationship to set the suitable upper barrier and lower barrier, which can ensure the fairness between the users and suppliers of the capitals.

4. Conclusion

By designing such a venture capital contract with double barriers and assuming the company value follows Kou's model, we have derived the pricing formula of the contract under the risk-neutral measure. We can set suitable upper and lower barriers to ensure that capitalists will obtain some protection while entrepreneurs will have enough opportunity to develop. There are some extensions to this paper: Firstly, the pricing formula we obtained is too complicated, we can aim at simplifying the formula by an approximation to the solution of the Laplace inversion; Secondly, since we obtain a theoretical pricing formula, we can make application into the real market to analyze its accuracy.

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